Chapter 8

Differential Equations

A function may be determined by a differential equation together with initial conditions.

In the first two sections of this chapter, we study two of the simplest and most important differential equations, which describe oscillations, growth, and decay. A variation of these equations leads to the hyperbolic functions, which are important for integration and other applications. To end the chapter, we study two general classes of differential equations whose solutions can be expressed in terms of integrals. These equations, called separable and linear equations, occur in a number of interesting geometrical and physical examples. We shall continue our study of differential equations in Chapter 12 after we have learned more calculus.

8.1 Oscillations

The solution of the equation for simple harmonic oscillations may be expressed in terms of trigonometric functions.

A common problem in physics is to determine the motion of a particle in a given force field. For a particle moving on a line, the force field is given by specifying the force F as a function of the position x and time t. The problem is to write x as a function of the time t so that the equation

$$F = m\frac{d^2x}{dt^2} \qquad \text{(Force = Mass} \times \text{Acceleration)} \tag{1}$$

is satisfied, where m is the mass of the particle. Equation (1) is called *Newton's second law of motion*.[1]

If the dependence of F on x and t is given, equation (1) becomes a *differential equation* in x—that is, an equation involving x and its derivatives with respect to t. It is called *second-order* since the second derivative of x appears. (If the second derivative of x were replaced by the first derivative, we would obtain a *first-order differential equation*—these are studied in the following sections). A *solution* of equation (1) is a function $x = f(t)$ which satisfies equation (1) for all t when $f(t)$ is substituted for x.

[1] Newton always expressed his laws of motion in words. The first one to formulate Newton's laws carefully as differential equations was L. Euler around 1750. (See C. Truesdell, *Essays on the History of Mechanics*, Springer-Verlag, 1968.) In what follows we shall not be concerned with specific units of measure for force—often it is measured in *newtons* (1 newton = 1 kilogram-meter per second^2). Later, in Section 9.5, we shall pay a little more attention to units.

For example, if the force is a *constant* F_0 and we rewrite equation (1) as

$$\frac{d^2x}{dt^2} = \frac{F_0}{m},$$

we can use our knowledge of antiderivatives to conclude that

$$\frac{dx}{dt} = \frac{F_0}{m}t + C_1$$

and

$$x = \frac{1}{2}\frac{F_0}{m}t^2 + C_1 t + C_2,$$

where C_1 and C_2 are constants. We see that the position of a particle moving in a constant force field is a quadratic function of time (or a linear function, if the force is zero). Such a situation occurs for vertical motion under the force of gravity near the earth's surface. More generally, if the force is a given function of t, independent of x, we can find the position as a function of time by integrating twice and using the initial position and velocity to determine the constants of integration.

In many problems of physical interest, though, the force is given as a function of *position* rather than time. One says that there is a (time-independent) force *field*, and that the particle feels the force given by the value of the field at the point where the particle happens to be.[2] For instance, if x is the downward displacement from equilibrium of a weight on a spring, then *Hooke's law* asserts that

$$F = -kx, \tag{2}$$

where k is a positive constant called the *spring constant*. (See Fig. 8.1.1.) This law, discovered experimentally, is quite accurate if x is not too large. There is

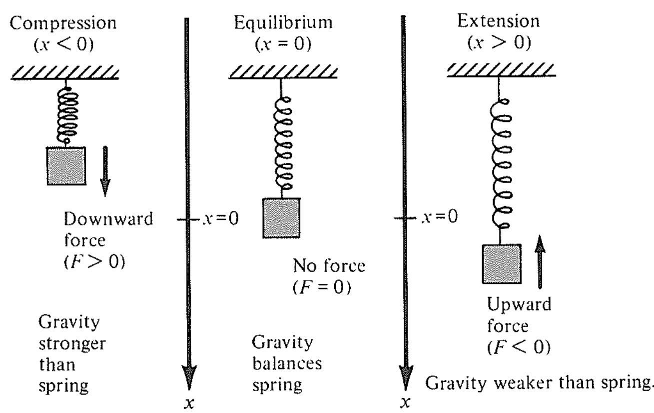


Figure 8.1.1. The force on a weight on a spring is proportional to the displacement from equilibrium.

a minus sign in the formula for F because the force, being directed toward the equilibrium, has the opposite sign to x. Substituting formula (2) into Newton's law (1) gives

$$-kx = m\frac{d^2x}{dt^2} \quad \text{or} \quad \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x.$$

It is convenient to write the ratio k/m as ω^2, where $\omega = \sqrt{k/m}$ is a new constant. This substitution gives us the *spring equation*:

[2] An example of a physical problem in which F depends on *both* x and t is the motion of a charged particle in a time-varying electric or magnetic field—see Exercise 13, Section 14.7.

$$\frac{d^2x}{dt^2} = -\omega^2 x. \tag{3}$$

Since x is an unknown function of t, we cannot find dx/dt by integrating the right-hand side. (In particular, dx/dt is *not* $-\frac{1}{2}\omega^2x^2 + C$, since it is t rather than x which is the independent variable.) Instead, we shall begin by using trial and error.

A good first guess, guided by the observation that weights on springs bob up and down, is

$$x = \sin t.$$

Differentiating twice with respect to t, we get

$$\frac{d^2x}{dt^2} = -\sin t = -x.$$

The factor ω^2 is missing, so we may be tempted to try $x = \omega^2 \sin t$. In this case, we get

$$\frac{d^2x}{dt^2} = -\omega^2 \sin t,$$

which is again $-x$. To bring out a *new factor* when we differentiate, we must take advantage of the chain rule. If we set $x = \sin \omega t$, then

$$\frac{dx}{dt} = \cos \omega t \frac{d(\omega t)}{dt} = \omega \cos \omega t$$

and

$$\frac{d^2x}{dt^2} = -\omega^2 \sin \omega t = -\omega^2 x,$$

which is just what we wanted. Looking back at our wrong guesses suggests that it would not hurt to put a constant factor in front, so that

$$x = B \sin \omega t$$

is also a solution for any B. Finally, we note that $\cos \omega t$ is another solution. In fact, if A and B are any two constants, then

$$x = A \cos \omega t + B \sin \omega t \tag{4}$$

is a solution of the spring equation (3), as you may verify by differentiating (4) twice. We say that the solution (4) is a *superposition* of the two solutions $A \sin \omega t$ and $B \cos \omega t$.

Example 1 Let $x = f(t) = A \cos \omega t + B \sin \omega t$. Show that x is periodic with period $2\pi/\omega$; that is, $f(t + 2\pi/\omega) = f(t)$.

Solution Substitute $t + 2\pi/\omega$ for t:

$$\begin{aligned} f\left(t + \frac{2\pi}{\omega}\right) &= A \cos\left[\omega\left(t + \frac{2\pi}{\omega}\right)\right] + B \sin\left[\omega\left(t + \frac{2\pi}{\omega}\right)\right] \\ &= A \cos[\omega t + 2\pi] + B \sin[\omega t + 2\pi] \\ &= A \cos \omega t + B \sin \omega t = f(t). \end{aligned}$$

Here we used the fact that the sine and cosine functions are themselves periodic with period 2π. ▲

The constants A and B are similar to the constants which arise when antiderivatives are taken. For any value of A and B, we have a solution. If we assign particular values to A and B, we get a *particular solution*. The choice of particular values of A and B is often determined by specifying *initial conditions*.

Example 2 Find a solution of the spring equation $d^2x/dt^2 = -\omega^2 x$ for which $x = 1$ and $dx/dt = 1$ when $t = 0$.

Solution In the solution $x = A\cos\omega t + B\sin\omega t$, we have to find A and B. Now $x = A$ when $t = 0$, so $A = 1$. Also $dx/dt = \omega B\cos\omega t - \omega A\sin\omega t = \omega B$ when $t = 0$. To make $dx/dt = 1$, we choose $B = 1/\omega$, and so the required solution is $x = \cos\omega t + (1/\omega)\sin\omega t$. ▲

In general, if we are given the initial conditions that $x = x_0$ and $dx/dt = v_0$ when $t = 0$, then

$$x = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t \tag{5}$$

is the unique function of the form (4) which satisfies these conditions.

Example 3 Solve for x: $d^2x/dt^2 = -x$, $x = 0$ and $dx/dt = 1$ when $t = 0$.

Solution Here $x_0 = 0$, $v_0 = 1$, and $\omega = 1$, so $x = x_0\cos\omega t + (v_0/\omega)\sin\omega t = \sin t$. ▲

Physicists expect that the motion of a particle in a force field is completely determined once the initial values of position and velocity are specified. Our solution of the spring equation will meet the physicists' requirements if we can show that *every* solution of the spring equation (3) is of the form (4). We turn to this task next.

In deriving formula (5) we saw that there are enough solutions of the form (4) so that x and dx/dt can be specified arbitrarily at $t = 0$. Thus if $x = f(t)$ is any solution of equation (3), then the function $g(t) = f(0)\cos\omega t + [f'(0)/\omega]\sin\omega t$ is a solution of the special form (4) with the same initial conditions as f: $g(0) = f(0)$ and $g'(0) = f'(0)$. We will now show that $f(t) = g(t)$ for all t by using the following fact: if $h(t)$ is any solution of equation (3), then the quantity $E = \frac{1}{2}\{[h'(t)]^2 + [\omega h(t)]^2\}$ is constant over time. This expression is called the *energy* of the solution h. To see that E is constant over time, we differentiate using the chain rule:

$$\frac{dE}{dt} = h'(t)h''(t) + \omega^2 h(t)h'(t) = h'(t)\{h''(t) + \omega^2 h(t)\}. \tag{6}$$

This equals zero since $h'' + \omega^2 h = 0$; thus E is constant over time. Now if f and g are solutions of equation (3) with $f(0) = g(0)$ and $f'(0) = g'(0)$, then $h(t) = f(t) - g(t)$ is also a solution with $h(0) = 0$ and $h'(0) = 0$. Thus the energy $E = \frac{1}{2}\{[h'(t)]^2 + [\omega h(t)]^2\}$ is constant; but it vanishes at $t = 0$, so it is identically zero. Thus, since two non-negative numbers which add to zero must both be zero: $h'(t) = 0$ and $\omega h(t) = 0$. In particular, $h(t) = 0$, and so $f(t) = g(t)$ as required.

The solution (4) of the spring equation can also be expressed in the form

$$x = \alpha\cos(\omega t - \theta),$$

where α and θ are constants. In fact, the addition formula for cosine gives

$$\alpha\cos(\omega t - \theta) = \alpha\cos\omega t\cos\theta + \alpha\sin\omega t\sin\theta. \tag{7}$$

This will be equal to $A\cos\omega t + B\sin\omega t$ if

$$\alpha\cos\theta = A \quad \text{and} \quad \alpha\sin\theta = B.$$

Thus α and θ must be the polar coordinates of the point whose cartesian coordinates are (A, B), and so we can always find such an α and θ with $\alpha \geqslant 0$. The form (7) is convenient for plotting, as shown in Fig. 8.1.2.

In Fig. 8.1.2 notice that the solution is a cosine curve with *amplitude* α which is shifted by the *phase shift* θ/ω. The number ω is called the *angular*

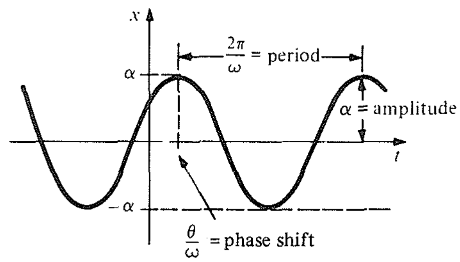


Figure 8.1.2. The graph of $x = \alpha\cos(\omega t - \theta)$.

frequency, since it is the time rate of change of the "angle" $\omega t - \theta$ at which the cosine is evaluated. The number of oscillations per unit time is the *frequency* $\omega/2\pi$ ($= 1/\text{period}$).

The motion described by the solutions of the spring equation is called *simple harmonic motion*. It arises whenever a system is subject to a restoring force proportional to its displacement from equilibrium. Such oscillatory systems occur in physics, biology, electronics, and chemistry.

Simple Harmonic Motion

Every solution of the *spring equation*

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \text{has the form} \qquad x = A\cos\omega t + B\sin\omega t,$$

where A and B are constants.

The solution can also be written

$$x = \alpha\cos(\omega t - \theta),$$

where (α,θ) are the polar coordinates of (A,B). [This function is graphed in Fig. 8.1.2.]

If the values of x and dx/dt are specified to be x_0 and v_0 at $t = 0$, then the unique solution is

$$x = x_0\cos\omega t + (v_0/\omega)\sin\omega t.$$

Example 4 Sketch the graph of the solution of $d^2x/dt^2 + 9x = 0$ satisfying $x = 1$ and $dx/dt = 6$ when $t = 0$.

Solution Using (5) with $\omega = 3$, $x_0 = 1$, and $v_0 = 6$, we have

$$x = \cos(3t) + 2\sin(3t) = \alpha\cos(3t - \theta).$$

Since $(A,B) = (1,2)$, and (α,θ) are its polar coordinates,

$$\alpha = \sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.2$$

and

$$\theta = \tan^{-1}2 \approx 1.1 \text{ radians (or } 63^\circ),$$

so $\theta/\omega \approx 0.37$. The period is $\tau = 2\pi/\omega \approx 2.1$. Thus we can plot the graph as shown in Fig. 8.1.3. ▲

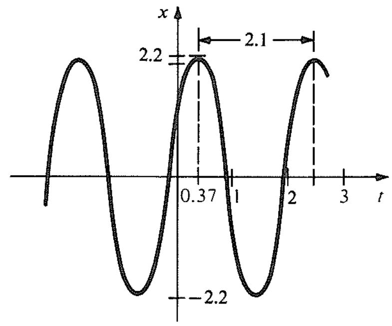


Figure 8.1.3. The graph of $x = 2.2\cos(3t - 1.1)$.

As usual, the independent variable need not always be called t, nor does the dependent variable need to be called x.

Example 5 (a) Solve for y: $d^2y/dx^2 + 9y = 0$, $y = 1$ and $dy/dx = -1$ when $x = 0$.
(b) Sketch the graph of y as a function of x.

Solution (a) Here $y_0 = 1$, $v_0 = -1$, and $\omega = \sqrt{9} = 3$ (using x in place of t and y in place of x), so

$$y = y_0 \cos \omega x + \frac{v_0}{\omega} \sin \omega x = \cos 3x - \tfrac{1}{3} \sin 3x.$$

(b) The polar coordinates of $(1, -\frac{1}{3})$ are given by $\alpha = \sqrt{1 + 1/9} = \sqrt{10}/3 \approx 1.05$ and $\theta = \tan^{-1}(-\frac{1}{3}) \approx -0.32$ (or $-18°$). Hence $y = \alpha\cos(\omega x - \theta)$ becomes $y = 1.05\cos(3x + 0.32)$, which is sketched in Fig. 8.1.4. Here $\theta/\omega = -0.1$ and $2\pi/\omega = 2.1$. ▲

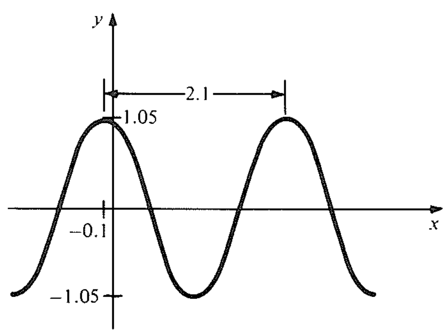


Figure 8.1.4. The graph of $y = (1.05)\cos(3x + 0.32)$.

A Remark on Notation. Up until now we have distinguished *variables*, which are mathematical objects that represent "quantities," and *functions*, which represent relations between quantities. Thus, when $y = f(x)$, we have written $f'(x)$ and dy/dx but *not* y', df/dx, or $y(x)$. It is common in mathematical writing to use the same symbol to denote a function and its dependent variable; thus one sometimes writes $y = y(x)$ to indicate that y is a function of x and then writes "$y' = dy/dx$," "$y(3)$ is the value of y when $x = 3$," and so on. Beginning with the next example, we will occasionally drop our scruples in distinguishing functions from variables and will use this abbreviated notation.

Example 6 Let M be a weight with mass 1 gram on a spring with spring constant $\frac{3}{2}$. Let the weight be initially extended by a distance of 1 centimeter moving at a velocity of 2 centimeters per second.

(a) How fast is M moving at $t = 3$?
(b) What is M's acceleration at $t = 4$?
(c) What is M's maximum displacement from the rest position? When does it occur?
(d) Sketch a graph of the solution.

Solution Let $x = x(t)$ denote the position of M at time t. We use the spring equation (3) with $\omega = \sqrt{k/m}$, where k is the spring constant and m is the mass of M. Since $k = \frac{3}{2}$ and $m = 1$, ω is $\sqrt{3/2}$. At $t = 0$, M is extended by a distance of 1 centimeter and moving at a velocity of 2 centimeters per second, so $x_0 = 1$ and $v_0 = 2$.

Now we have all the information we need to solve the spring equation. Applying formula (5) gives

$$x(t) = \cos\sqrt{3/2}\,t + \frac{2}{\sqrt{3/2}}\sin\sqrt{3/2}\,t.$$

(a) $x'(t) = -\sqrt{3/2}\sin\sqrt{3/2}\,t + 2\cos\sqrt{3/2}\,t$. Substituting $t = 3$ gives

$$x'(3) = -\sqrt{3/2}\sin 3\sqrt{3/2} + 2\cos 3\sqrt{3/2} \approx -1.1 \text{ centimeters per second.}$$

(Negative velocity represents upward motion.)

(b) $$x''(t) = \frac{d}{dt}\left(-\sqrt{\frac{3}{2}}\sin\sqrt{\frac{3}{2}}\,t + 2\cos\sqrt{\frac{3}{2}}\,t\right) = -\frac{3}{2}\cos\sqrt{\frac{3}{2}}\,t - \sqrt{6}\sin\sqrt{\frac{3}{2}}\,t$$

and thus the acceleration at $t = 4$ is

$$x''(4) = -\tfrac{3}{2}\cos 2\sqrt{6} - \sqrt{6}\sin 2\sqrt{6} \approx 2.13 \text{ centimeters per second}^2$$

(c) The simplest way to find the maximum displacement is to use the "phase-amplitude" form (7). The maximum displacement is the amplitude $\alpha = \sqrt{A^2 + B^2}$, where A and B are the coefficients of sin and cos in the solution. Here, $A = 1$ and $B = 2/\sqrt{3/2}$, so $\alpha^2 = 1 + 4/(3/2) = 1 + 8/3 = 11/3$, so $\alpha = \sqrt{11/3} \approx 1.91$ centimeters, which is a little less than twice the initial displacement.

(d) To sketch a graph we also need the phase shift. Now $\theta = \tan^{-1}(B/A)$, which is in the first quadrant since A and B are positive. Thus $\theta = \tan^{-1}(2/\sqrt{3/2}) \approx 1.02$, and so the maximum point on the graph (see Fig. 8.1.2) occurs at $\theta/\omega = 1.02/\sqrt{3/2} \approx 0.83$. The period is $2\pi/\omega \approx 5.13$. The graph is shown in Fig. 8.1.5. ▲

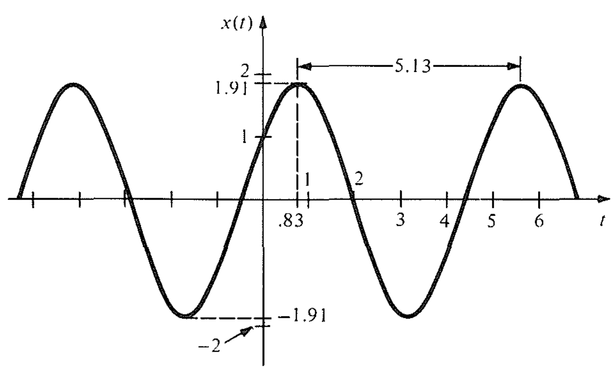


Figure 8.1.5. The graph of $x(t) = \cos\sqrt{\frac{3}{2}}\,t + \sqrt{\frac{8}{3}}\sin\sqrt{\frac{3}{2}}\,t \approx 1.91\cos(\sqrt{\frac{3}{2}}\,t - 1.02)$.

Supplement to Section 8.1: Linearized Oscillations

The spring equation can be applied to determine the *approximate* motion of *any* system subject to a restoring force, even if the force is not linear in the

displacement. Such forces occur in more realistic models for springs and in equations for electric circuits. Suppose that we wish to solve the equation of motion

$$m\frac{d^2x}{dt^2}=f(x), \tag{8}$$

where the force function $f(x)$ satisfies the conditions: (i) $f(x_0)=0$; and (ii) $f'(x_0)<0$, for some position x_0. The point x_0 is an *equilibrium* position since the constant function $x(t)=x_0$ satisfies the equation of motion, by condition (i). By condition (ii), the force is positive when x is near x_0 and $x<x_0$, while the force is negative when x is near x_0 and $x>x_0$. Thus the particle is being pushed back toward x_0 whenever it is near that point, just as with the spring in Figure 8.1.1.

Rather than trying to solve equation (8) directly, we shall replace $f(x)$ by its linear approximation $f(x_0)+f'(x_0)(x-x_0)$ at x_0. Since $f(x_0)=0$, the equation (8) becomes

$$m\frac{d^2x}{dt^2}=f'(x_0)(x-x_0), \tag{8$'$}$$

which is called the *linearization* of equation (8) at x_0. If we write k for the positive number $-f'(x_0)$ and $y=x-x_0$ for the displacement from equilibrium, then we get

$$m\frac{d^2y}{dt^2}=-ky,$$

which is precisely the spring equation.

We thus conclude that, to the degree that the linear approximation of the force is valid, the particle oscillates around the equilibrium point x_0 with period $2\pi/\sqrt{-f'(x_0)/m}$. It can be shown that the particle subject to the exact force law (8) also oscillates around x_0, but with a period which depends upon the amplitude of the oscillations. As the amplitude approaches zero, the period approaches $2\pi/\sqrt{-f'(x_0)/m}$, which is the period for the linearized equation.

Here is an application of these ideas:

By decomposing the gravitational force on a pendulum of mass m and length l into components parallel and perpendicular to the pendulum's axis, it can be shown that the displacement angle θ of the pendulum from its equilibrium (vertical) position satisfies the differential equation $m(d^2\theta/dt^2)=-m(g/l)\sin\theta$, where $g=9.8$ meters per second^2 is the gravitational constant. (See Fig. 8.1.6). The force function is $f(\theta)=-(mg/l)\sin\theta$. Since $f(0)=0$, $\theta=0$ is an equilibrium point. Since $f'(0)=-(mg/l)\sin'(0)=-mg/l$, the linearized equation is $m\,d^2\theta/dt^2=-(mg/l)\theta$. The period of oscillations for the linearized equation is thus $2\pi/\sqrt{(mg/l)m}=2\pi\sqrt{l/g}$. (See Review Exercise 83 for information on the solution of the nonlinear equation.)

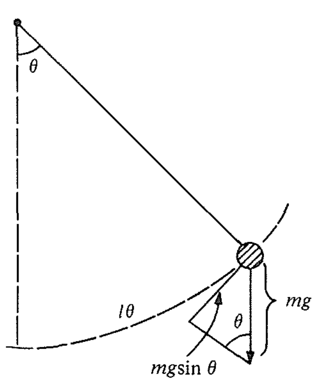


Figure 8.1.6. The forces acting on a pendulum.

A point x_0 satisfying the conditions above is called a *stable equilibrium point*. The word *stable* refers to the fact that motions which start near x_0 with small initial velocity stay near x_0.[3] If $f(x_0)=0$ but $f'(x_0)>0$, we have an *unstable* equilibrium point (see Section 8.3).

[3] We have only proved stability for the linearized equations. Using conservation of energy, one can show that the motion for the exact equations stays near x_0 as well. (See Exercise 33.)

Exercises for Section 8.1

1. Show that $f(t) = \cos(3t)$ is periodic with period $2\pi/3$.
2. Show that $f(t) = 8\sin(\pi t)$ is periodic with period 2.
3. Show that $f(t) = \cos(6t) + \sin(3t)$ is periodic with period $2\pi/3$.
4. Show that $f(t) = 3\sin(\pi t/2) + 8\cos(\pi t)$ is periodic with period 4.

In Exercises 5–8, find the solution of the given equation with the prescribed values of x and $x' = dx/dt$ at $t = 0$.

5. $x'' + 9x = 0$, $x(0) = 1$, $x'(0) = -2$.
6. $x'' + 16x = 0$, $x(0) = -1$, $x'(0) = -1$.
7. $x'' + 12x = 0$, $x(0) = 0$, $x'(0) = -1$.
8. $x'' + 25x = 0$, $x(0) = 1$, $x'(0) = 0$.

In Exercises 9–12, sketch the graph of the given function and find the period, amplitude, and phase shift.

9. $x = 3\cos(3t - 1)$.
10. $x = 2\cos(5t - 2)$.
11. $x = 4\cos(t + 1)$.
12. $x = 6\cos(3t + 4)$.

In Exercises 13–16, solve the given equation for x and sketch the graph.

13. $\dfrac{d^2x}{dt^2} + 4x = 0$; $x = -1$ and $\dfrac{dx}{dt} = 0$ when $t = 0$.
14. $\dfrac{d^2x}{dt^2} + 16x = 0$, $x = 1$ and $\dfrac{dx}{dt} = 0$ when $t = 0$.
15. $\dfrac{d^2x}{dt^2} + 25x = 0$, $x = 5$ and $\dfrac{dx}{dt} = 5$ when $t = 0$.
16. $\dfrac{d^2x}{dt^2} + 25x = 0$, $x = 5$ at $t = 0$, and $\dfrac{dx}{dt} = 5$ when $t = \pi/4$.
17. Find the solution of $\dfrac{d^2y}{dt^2} = -4y$ for which $y = 1$ and $\dfrac{dy}{dt} = 3$ when $t = 0$.
18. Find $y = f(x)$ if $f'' + 4f = 0$ and $f(0) = 0$, $f'(0) = -1$.
19. Suppose that $f(x)$ satisfies $f'' + 16f = 0$ and $f(0) = 2$, $f'(0) = 0$. Sketch the graph $y = f(x)$.
20. Suppose that $z = g(r)$ satisfies $9z'' + z = 0$ and $z(0) = -1$, $z'(0) = 0$. Sketch the graph $z = g(r)$.
21. A mass of 1 kilogram is hanging from a spring. If $x = 0$ is the equilibrium position, it is given that $x = 1$ and $dx/dt = 1$ when $t = 0$. The weight is observed to oscillate with a frequency of twice a second.
    (a) What is the spring constant?
    (b) Sketch the graph of x as a function of t, indicating the amplitude of the motion on your drawing.
22. An observer sees a weight of 5 grams on a spring undergoing the motion $x(t) = 6.1\cos(2t - \pi/6)$.
    (a) What is the spring constant?
    (b) What is the force acting on the weight at $t = 0$? At $t = 2$?
23. What happens to the frequency of oscillations if three equal masses are hung from a spring where there was one mass before?
24. Find a differential equation of the "spring" type satisfied by the function $y(t) = 3\cos(t/4) - \sin(t/4)$.

★25. A "flabby" spring exerts a force $f(x) = -3x + 2x^3$ when it is displaced a distance x from its equilibrium state, $x = 0$.
    (a) Write the equation of motion for an object of mass 27, vibrating on this spring.
    (b) Write the linearized equation of motion at $x_0 = 0$.
    (c) Find the period of linearized oscillations.

★26. (a) Find the equilibrium position of an object which satisfies the equation of motion

$$4\frac{d^2x}{dt^2} = -x^3 + x^2 - x + 1.$$

(b) What is the frequency of linearized oscillations?

★27. An atom of mass m in a linear molecule is subjected to forces of attraction by its neighbors given by

$$f(x) = k_1(x - x_1)^3 + k_2(x - x_2)^3,$$
$$k_1, k_2 > 0, \quad 0 < x_1 < x < x_2.$$

(a) Compute the equilibrium position.
(b) Show that motion near this equilibrium is unstable.

★28. The equation for a spring with friction is

$$m\frac{d^2x}{dt^2} = -kx - \delta\frac{dx}{dt}$$

(spring equation with damping).
(a) If $\delta^2 < 4km$, check that a solution is

$$x(t) = e^{-\delta t/2m}(A\cos\omega t + B\sin\omega t),$$

where $\omega^2 = k/m - \delta^2/4m^2 > 0$.
(b) Sketch the general appearance of the graph of the solution in (a) and define the "period" of oscillation.
(c) If the force $-kx$ is replaced by a function $f(x)$ satisfying $f(0) = 0$, $f'(0) < 0$, find a formula for the frequency of damped linearized oscillations.

★29. Suppose that $x = f(t)$ satisfies the spring equation. Let $g(t) = at + b$, where a and b are constants. Show that if the composite function $f \circ g$ satisfies the spring equation (with the same ω), then $a = \pm 1$. What about b?

★30. (a) Suppose that $f(t)$ is given and that $y = g(t)$ satisfies $d^2y/dt^2 + \omega^2 y = f(t)$. Show that

$x = y + A\sin\omega t + B\cos\omega t$ represents the general solution of $d^2x/dt^2 + \omega^2 x = f$; that is, x is a solution and any solution has this form. One calls y a *particular* solution and x the *general* solution.

(b) Solve $d^2x/dt^2 + \omega^2 x = k$ if $x = 1$ and $dx/dt = -1$ when $t = 0$; k is a nonzero constant.

(c) Solve $d^2x/dt^2 + \omega^2 x = \omega^2 t$ if $x = -1$ and $dx/dt = 3$ when $t = 0$.

Exercises 31 and 32 outline the complete proof of the following theorem using the "method of variation of constants": *Let $x = f(t)$ be a twice-differentiable function of t such that $(d^2x/dt^2) + \omega^2 x = 0$. Then $x = A\cos\omega t + B\sin\omega t$ for constants A and B.*

★31. Some preliminary calculations are done first. Write

$$x = A(t)\cos\omega t + B(t)\sin\omega t. \tag{9}$$

It is possible to choose $A(t)$ and $B(t)$ in many ways, since for each t either $\sin\omega t$ or $\cos\omega t$ is nonzero. To determine $A(t)$ and $B(t)$ we add a second equation:

$$\frac{dx}{dt} = -\omega A(t)\sin\omega t + \omega B(t)\cos\omega t. \tag{10}$$

This equation is obtained by differentiating (9) *pretending* that $A(t)$ and $B(t)$ are constants. Since this is what we are trying to prove, we should be very suspicious here of circular reasoning. But push on and see what happens. Show that

$$B(t) = x\sin\omega t + \frac{dx/dt}{\omega}\cos\omega t. \tag{11}$$

Similarly, show that

$$A(t) = x\cos\omega t - \frac{dx/dt}{\omega}\sin\omega t. \tag{12}$$

★32. Use the calculations in Exercise 31 to give the proof of the theorem, making sure to avoid circular reasoning. We are given $x = f(t)$ and ω such that $(d^2x/dt^2) + \omega^2 x = 0$. Define $A(t)$ and $B(t)$ by equations (11) and (12). Show that $A(t)$ and $B(t)$ are in fact constants by differentiating (11) and (12) to show that $A'(t)$ and $B'(t)$ are identically zero. Then rewrite formulas (11) and (12) as

$$B = x(t)\sin\omega t + \frac{dx/dt}{\omega}\cos\omega t, \tag{13}$$

and

$$A = x(t)\cos\omega t - \frac{dx/dt}{\omega}\sin\omega t. \tag{14}$$

Use these formulas to show $A\cos\omega t + B\sin\omega t = x$, which proves the theorem.

★33. Suppose that $m(d^2x/dt^2) = f(x)$, where $f(x_0) = 0$ and $f'(x_0) < 0$. Let $V(x)$ be an antiderivative of $-f$.

(a) Show that x_0 is a local minimum of V.

(b) Show that $dE/dt = 0$, where the energy E is given by $E = \frac{1}{2}m(dx/dt)^2 + V(x)$.

(c) Use conservation of energy from (b) to show that if dx/dt and $x - x_0$ are sufficiently small at $t = 0$, then they both remain small.

8.2 Growth and Decay

The solution of the equation for population growth may be expressed in terms of exponential functions.

Many quantities, such as bank balances, populations, the radioactivity of ores, and the temperatures of hot objects change at a rate which is proportional to the current value of the quantity. In other words, if $f(t)$ is the quantity at time t, then f satisfies the differential equation

$$f'(t) = \gamma f(t), \tag{1}$$

where γ is a constant. For example, in the specific case of temperature, it is an experimental fact that the temperature of a hot object decreases at a rate proportional to the difference between the temperature of the object and that of its surroundings. This is called *Newton's law of cooling*.

Example 1 The temperature of a hot bowl of porridge decreases at a rate 0.0837 times the difference between its present temperature and room temperature (fixed at 20°C). Write down a differential equation for the temperature of the porridge. (Time is measured in minutes and temperature in °C.)

Solution Let T be the temperature (°C) of the porridge and let $f(t) = T - 20$ be its temperature above 20°C. Then $f'(t) = dT/dt$ and so

$$f'(t) = -(0.0837)f(t)$$

i.e.,

$$\frac{dT}{dt} = -(0.0837)(T - 20).$$

The minus sign is used because the temperature is *decreasing* when T is greater than 20; $\gamma = -0.0837$. ▲

We solve equation (1) by guesswork, just as we did the spring equation. The answer must be a function which produces itself times a constant when differentiated once. It is reasonable that such a function should be related to the exponential since e^t has the reproductive property $(d/dt)e^t = e^t$. To get a factor γ, we replace t by γt. Then $(d/dt)e^{\gamma t} = \gamma e^{\gamma t}$, by the chain rule. We can also insert a constant factor A to get

$$\frac{d}{dt}(Ae^{\gamma t}) = \gamma(Ae^{\gamma t}).$$

Thus $f(t) = Ae^{\gamma t}$ solves equation (1). If we pick $t = 0$, we see that $A = f(0)$. This gives us a solution of equation (1); we shall show below that it is the only solution.

The Solution of $f' = \gamma f$

Given $f(0)$, there is one and only one solution to the differential equation $f'(t) = \gamma f(t)$, namely

$$f(t) = f(0)e^{\gamma t} \tag{2}$$

To show that formula (2) gives the *only* solution, let us suppose that $g(t)$ also satisfies $g'(t) = \gamma g(t)$ and $g(0) = f(0)$. We will show that $g(t) = f(0)e^{\gamma t}$. To do this, consider the quotient

$$h(t) = \frac{g(t)}{e^{\gamma t}} = e^{-\gamma t}g(t)$$

and differentiate:

$$h'(t) = -\gamma e^{-\gamma t}g(t) + e^{-\gamma t}g'(t) = -\gamma e^{-\gamma t}g(t) + \gamma e^{-\gamma t}g(t) = 0.$$

Since $h'(t) = 0$, we may conclude that h is constant; but $h(0) = e^{-0}g(0) = f(0)$, so $e^{-\gamma t}g(t) = h(t) = f(0)$, and thus

$$g(t) = f(0)e^{\gamma t} = f(t),$$

as required.

Example 2 If $dx/dt = 3x$, and $x = 2$ at $t = 0$, find x for all t.

Solution If $x = f(t)$, then $f(0) = 2$ and $f' = 3f$, so $\gamma = 3$ in the box above. Hence, by formula (2), $x = f(t) = 2e^{3t}$. ▲

Example 3 Find a formula for the temperature of the bowl of porridge in Example 1 if it starts at 80°C. Jane Cool refuses to eat the porridge when it is too cold—namely, if it falls below 50°C. How long does she have to come to the table?

Solution Let $f(t) = T - 20$ as in Example 1. Then $f'(t) = -0.0837f(t)$ and the initial condition is $f(0) = 80 - 20 = 60$. Therefore

$$f(t) = 60e^{-0.0837t}.$$

Hence $T = f(t) + 20 = 60e^{-0.0837t} + 20$. When $T = 50$, we have

$$\begin{aligned} 50 &= 60e^{-0.0837t} + 20 \\ 1 &= 2e^{-0.0837t} \\ e^{0.0837t} &= 2 \\ 0.0837t &= \ln 2 = 0.693. \end{aligned}$$

Thus $t = 8.28$ minutes. Jane has a little more than 8 minutes before the temperature drops to 50°C. ▲

Note how the behavior of solution (2) depends on the sign of γ. If $\gamma > 0$, then $e^{\gamma t} \to \infty$ as $t \to \infty$ (growth); if $\gamma < 0$, then $e^{\gamma t} \to 0$ as $t \to \infty$ (decay). See Fig. 8.2.1.

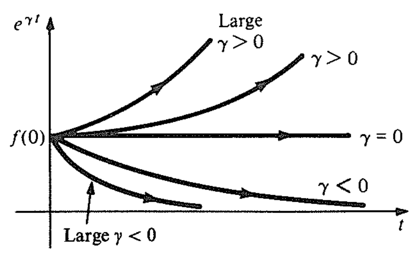


Figure 8.2.1. Growth occurs if $\gamma > 0$, decay if $\gamma < 0$.

A quantity which depends on time according to equation (1) (or, equivalently, (2)) is said to undergo *natural growth* or *decay*.

Natural Growth or Decay

The solution of $f' = \gamma f$ is $f(t) = f(0)e^{\gamma t}$ which grows as t increases if $\gamma > 0$ and which decays as t increases if $\gamma < 0$.

Example 4 Suppose that $y = f(x)$ satisfies $dy/dx + 3y = 0$ and $y = 2$ at $x = 0$. Sketch the graph $y = f(x)$.

Solution The equation may be written $dy/dx = -3y$ which has the form of equation (1) with $\gamma = -3$ and the independent variable t replaced by x. By formula (2) the solution is $y = 2e^{-3x}$. The graph is sketched in Fig. 8.2.2. ▲

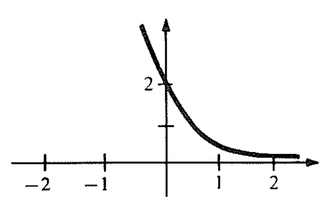


Figure 8.2.2. The graph of $y = 2e^{-3x}$.

If a quantity $f(t)$ is undergoing natural growth or decay, i.e., $f(t) = f(0)e^{\gamma t}$, we notice that

$$\frac{f(t+s)}{f(t)} = \frac{f(0)e^{\gamma(t+s)}}{f(0)e^{\gamma t}} = e^{\gamma s} = \frac{f(s)}{f(0)},$$

so

$$\frac{f(t+s)}{f(t)} = \frac{f(s)}{f(0)}. \tag{3}$$

Thus the percentage increase or decrease in f over a time interval of length s is fixed, independent of when we start. This property, characteristic of natural growth or decay is called *uniform growth* or *decay*. It states, for example, that if you leave money in a bank with a fixed interest rate, then the percentage increase in your balance over each period of a given length (say 3 years) is the same.

We can show that if f undergoes uniform growth (or decay), then f undergoes natural growth (or decay). Indeed, write equation (3) as

$$f(t+s) = \frac{f(t)f(s)}{f(0)}$$

and differentiate with respect to s:

$$f'(t+s) = \frac{f(t)f'(s)}{f(0)}.$$

Now set $s = 0$ and let $\gamma = f'(0)/f(0)$:

$$f'(t) = \gamma f(t),$$

which is the law of natural growth. Thus natural growth and uniform growth are equivalent notions.

We shall now discuss *half-life problems*. It is a physical law that radioactive substances decay at a rate proportional to the amount of the substance present. If $f(t)$ denotes the amount of the substance at time t, then the physical law states that $f'(t) = -\kappa f(t)$ for a positive constant κ. (The *minus* sign is inserted since the substance is *decaying*.) Thus, formula (2) with $\gamma = -\kappa$ gives $f(t) = f(0)e^{-\kappa t}$. The *half-life* $t_{1/2}$ is the time required for half the substance to remain. Therefore $f(t_{1/2}) = \frac{1}{2}f(0)$, so $f(0)e^{-\kappa t_{1/2}} = \frac{1}{2}f(0)$. Hence $2 = e^{\kappa t_{1/2}}$, so

$$t_{1/2} = (1/\kappa)\ln 2. \tag{4}$$

Half-Life

If a quantity decays according to the law $f'(t) = -\kappa f(t)$, it will be half gone after the elapse of time $t_{1/2} = (1/\kappa)\ln 2$; $t_{1/2}$ is called the *half-life*.

Example 5 Radium decreases at a rate of 0.0428% per year. What is its half-life?

Solution *Method 1*. We use the preceding box. Here $\kappa = -0.000428$, so the half-life is $t_{1/2} = (\ln 2)/0.000428 \approx 1620$ years.

Method 2. It is efficient in many cases to rederive the formula for half-life rather than memorizing it. With this approach, the solution looks like this: Let $f(t)$ denote the amount of radium at time t. We have $f'(t) = -0.000428f(t)$, so $f(t) = f(0)e^{-0.000428t}$. If $f(t) = \frac{1}{2}f(0)$, then $\frac{1}{2} = e^{-0.000428t}$; that is, $e^{0.000428t} = 2$, or $0.000428t = \ln 2$. Hence, $t = (\ln 2)/0.000428 \approx 1620$ years. ▲

Example 6 A certain radioactive substance has a half-life of 5085 years. What percentage will remain after an elapse of 10,000 years?

Solution If $f(t)$ is the amount of the substance after an elapse of time t, then $f(t) = f(0)e^{-\kappa t}$ for a constant κ. Since the half-life is 5085, $\frac{1}{2} = e^{-5085\kappa}$, i.e., $\kappa = (1/5085)\ln 2$. The amount after time $t = 10{,}000$ is

$$f(t) = f(0)e^{-10{,}000\kappa} = f(0)e^{-10{,}000 \ln 2/5085} = 0.256\, f(0).$$

Thus 25.6% remains. ▲

Another quantity which often changes at a rate proportional to the amount present is a population.

Example 7 The population of the planet $\delta o o \mu$ is increasing at an instantaneous rate of 5% per year. How long will it take for the population to double?

Solution Let $P(t)$ denote the population. Since the rate of increase is 5%, $P'(t) = 0.05P(t)$, so $P(t) = P(0)e^{0.05t}$. In order for $P(t)$ to be $2P(0)$, we should have $2 = e^{0.05t}$; that is, $0.05t = \ln 2$ or $t = 20\ln 2$. Using $\ln 2 = 0.6931$, we get $t \approx 13.862$ years. ▲

In this and similar examples, there is the possibility of confusion over the meaning of phrases like "increases at a rate of 5% per year." When the word "instantaneous" is used, it means that the *rate* is 5%, i.e., $P' = 0.05P$. This does *not* mean that after one year the population has increased by 5%—it will in fact be greater than that.

In Section 6.4 we saw the distinction between annual rates and instantaneous rates in connection with problems of finance. If an initial principal P_0 is left in an account earning r% compounded continuously, this means that the amount of money P in the account at time t changes according to

$$\frac{dP}{dt} = \frac{r}{100}P.$$

Thus, by formula (2),

$$P(t) = e^{rt/100}P_0. \tag{5}$$

The *annual percentage rate* is the percentage increase after one year, namely

$$100\left(\frac{P(1) - P_0}{P_0}\right) = 100(e^{r/100} - 1). \tag{6}$$

This agrees with formula (8) derived in Section 6.4 by a different method.

Example 8 How long does it take for a quantity of money to triple if it is left in an account earning 8.32% interest compounded continuously?

Solution Let P_0 be the amount deposited. By formula (5),

$$P(t) = e^{0.0832t}P_0.$$

If $P(t) = 3P_0$, then

$$3 = e^{0.0832t},$$
$$0.0832t = \ln 3,$$
$$t = \frac{\ln 3}{0.0832} \approx 13.2 \text{ years. ▲}$$

Exercises for Section 8.2

1. The temperature T of a hot iron decreases at a rate 0.11 times the difference between its present temperature and room temperature (20°C). If time is measured in minutes, write a differential equation for the temperature of the iron.
2. A population P of monkeys increases at a rate 0.051 per year times the current population. Write down a differential equation for P.
3. The amount Q in grams of a radioactive substance decays at a rate 0.00028 per year times the current amount present. Write a differential equation for Q.
4. The amount M of money in a bank increases at an instantaneous rate of 13.51% per year times the present amount. Write a differential equation for M.

Solve the differential equations in Exercises 5–12 using the given data.

5. $f' = -3f$, $f(0) = 2$.
6. $\frac{dx}{dt} = x$, $x = 3$ when $t = 0$.
7. $\frac{dx}{dt} - 3x = 0$, $x = 1$ when $t = 0$.
8. $\frac{du}{dr} - 13u = 0$, $u = 1$ when $r = 0$.
9. $\frac{dy}{dt} = 8y$, $y = 2$ when $t = 1$.
10. $\frac{dy}{dx} = -10y$, $y = 1$ when $x = 1$.
11. $\frac{dv}{ds} + 2v = 0$, $v = 2$ when $s = 3$.
12. $\frac{dw}{dx} + aw = 0$, $w = b$ when $x = c$ (a, b, c constants).
13. If the iron in Exercise 1 starts out at 210°C, how long (in minutes) will it take for it to cool to 100°C?
14. If the population P in Exercise 2 starts out at $P(0) = 800$, how long will it take to reach 1500?
15. If $Q(0) = 1$ gram in Exercise 3, how long will it take until $Q = \frac{1}{2}$ gram?
16. How long does it take the money in Exercise 4 to double?

Solve each equation in Exercises 17–20 for $f(t)$ and sketch its graph.

17. $f' - 3f = 0$, $f(0) = 1$
18. $f' + 3f = 0$, $f(0) = 1$
19. $f' = 8f$, $f(0) = e$
20. $f' = 8f$, $f(1) = e$

Without solving, tell whether or not the solutions of the equations in Exercises 21–24 are increasing or decreasing.

21. $\frac{dx}{dt} = 3x$, $x = 1$ when $t = 0$.
22. $\frac{dx}{dt} = 3x$, $x = -1$ when $t = 0$.
23. $f' = -3f$, $f(0) = 1$.
24. $f' = -3f$, $f(0) = -1$.
25. A certain radioactive substance decreases at a rate of 0.0021% per year. What is its half-life?
26. Carbon-14 decreases at a rate of 0.01238% per year. What is its half-life?
27. It takes 300,000 years for a certain radioactive substance to decay to 30% of its original amount. What is its half-life?
28. It takes 80,000 years for a certain radioactive substance to decrease to 75% of its original amount. Find the half-life.
29. The half-life of uranium is about 0.45 billion years. If 1 gram of uranium is left undisturbed, how long will it take for 90% of it to have decayed?
30. The half-life of substance X is 3,050 years. What percentage of substance X remains after 12,200 years?
31. Carbon-14 is known to satisfy the decay law $Q = Q_0e^{-0.0001238t}$ for the amount Q present after t years. Find the age of a bone sample in which the carbon-14 present is 70% of the original amount Q_0.
32. Consider two decay laws for radioactive carbon-14: $Q = Q_0e^{-\alpha t}$, $Q = Q_0e^{-\beta t}$, where $\alpha = 0.0001238$ and $\beta = 0.0001236$. Find the percentage error between the two exponential laws for predicting the age of a skull sample with 50% of the carbon-14 decayed. (See Exercise 31.)
33. A certain bacterial culture undergoing natural growth doubles in size after 10 minutes. If the culture contains 100 specimens at time $t = 0$, when will the number have increased to 3000 specimens?
34. A rabbit population doubles in size every 18 months. If there are 10,000 rabbits at $t = 0$, when will the population reach 100,000?
35. A bathtub is full of hot water at 110°F. After 10 minutes it will be 90°F. The bathroom is at 65°F. George College refuses to enter water below 100°F. How long can he wait to get in the tub?
36. A blacksmith's hot iron is at 830°C in a room at 32°C. After 1 minute it is 600°C. The blacksmith has to wait until it reaches 450°C. How long does it take after the 600°C temperature is reached?
37. How long does it take for money left in an account earning $7\frac{1}{2}$% interest compounded continuously to quadruple?
38. In a certain bank account, money doubles in 10 years. What is the annual interest rate compounded continuously?

39. A credit card company advertises: "Your interest rate on the unpaid balance is 17% compounded continuously, but federal law requires us to state that your annual interest rate is 18.53%." Explain.
40. If a credit card charges an interest rate of 21% compounded continuously, what is the actual annual percentage rate?
41. A certain calculus textbook sells according to this formula: $S(t) = 2000 - 1000e^{-0.3t}$, where t is the time in years and $S(t)$ is the number of books sold.
    (a) Find $S'(t)$.
    (b) Find $\lim_{t\to\infty} S(t)$ and discuss.
    (c) Graph S.
42. A foolish king, on losing a famous bet, agrees to pay a wizard 1 cent on the first day of the month, 2 cents on the second day, 4 cents on the third, and so on, each day doubling the sum. How much is paid on the thirtieth day?
43. The author of a certain calculus textbook is awake writing in the stillness of 2 A.M. A sound disturbs him. He discovers that the toilet tank fills up fast at first, then slows down as the water is being shut off. Examining the insides of the tank and contemplating for a moment, he thinks that maybe during shutoff the rate of flow of water into the tank is proportional to the height left to go; that is, $dx/dt = c(h - x)$, where x = height of water, h = desired height of water, and c = a constant (depending on the mechanism). Show that $x = h - Ke^{-ct}$. What is K?
    Looking at this formula for x, he says "That explains why my tank is always filling!" and goes to bed.
44. (a) Verify that the solution of $dy/dt = p(t)y$ is $y = y_0 \exp P(t)$, where $P(t)$ is the antiderivative of $p(t)$ with $P(0) = 0$.
    (b) Solve $dy/dt = ty$; $y = 1$ when $t = 0$.

★45. (a) Show that a solution of $t(da/dt) = a + h$ is
$$a(t) = t\int_1^t \frac{h(s)}{s^2}\,ds + tC.$$
(b) Solve $t(da/dt) = a + e^{-1/t}$, $a(1) = 1$.

★46. Develop a general formula for the *doubling time* of a population in terms of its growth rate.

★47. Develop a general formula for the half-life of the amplitude of a damped spring (Exercise 28, Section 8.1).

8.3 The Hyperbolic Functions

The points $(\cos t, \sin t)$ *lie on a circle, and* $(\cosh t, \sinh t)$ *lie on a hyperbola.*

The hyperbolic functions are certain combinations of exponential functions which satisfy identities very similar to those for the trigonometric functions. We shall see in the next section that the inverse hyperbolic functions are important in integration.

A good way to introduce the hyperbolic functions is through a differential equation which they solve. Recall that $\sin t$ and $\cos t$ are solutions of the equation $d^2x/dt^2 + x = 0$. Now we switch the sign and consider $d^2x/dt^2 - x = 0$. (This corresponds to a negative spring constant!)

We already know one solution to this equation: $x = e^t$. Another is e^{-t}, because when we differentiate e^{-t} twice we bring down, via the chain rule, two minus signs and so recover e^{-t} again. The combination

$$x = Ae^t + Be^{-t}$$

is also a solution, as is readily verified.

If we wish to find a solution analogous to the sine function, with $x = 0$ and $dx/dt = 1$ when $t = 0$, we must pick A and B so that

$$0 = A + B,$$
$$1 = A - B,$$

so $A = \frac{1}{2}$ and $B = -\frac{1}{2}$, giving $x = (e^t - e^{-t})/2$.

Similarly, if we wish to find a solution analogous to the cosine function, we should pick A and B such that $x = 1$ and $dx/dt = 0$ when $t = 0$; that is,

$$1 = A + B,$$
$$0 = A - B,$$

so $A = B = \frac{1}{2}$, giving $x = (e^t + e^{-t})/2$.

This reasoning leads to the following definitions.

Hyperbolic Sine and Cosine

The *hyperbolic sine function*, written $\sinh t$, is defined by

$$\sinh t = \frac{e^t - e^{-t}}{2}. \tag{1}$$

The *hyperbolic cosine function*, written $\cosh t$, is defined by

$$\cosh t = \frac{e^t + e^{-t}}{2}. \tag{2}$$

(See Fig. 8.3.1.)

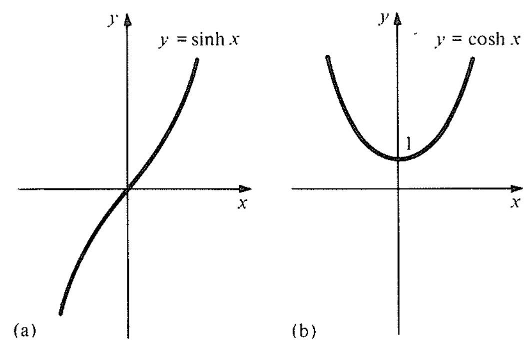


Figure 8.3.1. The graphs of $y = \sinh x$ and $y = \cosh x$.

The usual trigonometric functions $\sin t$ and $\cos t$ are called the *circular functions* because $(x, y) = (\cos t, \sin t)$ parametrizes the circle $x^2 + y^2 = 1$. The functions $\sinh t$ and $\cosh t$ are called *hyperbolic functions* because $(x, y) = (\cosh t, \sinh t)$ parametrizes one branch of the hyperbola $x^2 - y^2 = 1$; that is, for any t, we have the identity

$$\cosh^2 t - \sinh^2 t = 1. \tag{3}$$

(See Fig. 8.3.2.)

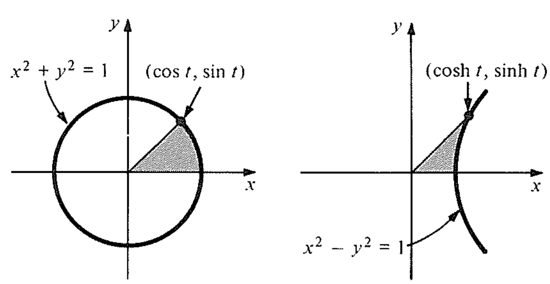


Figure 8.3.2. The points $(\cos t, \sin t)$ lie on a circle, while $(\cosh t, \sinh t)$ lie on a hyperbola.

To prove formula (3), we square formulas (1) and (2):

$$\cosh^2 t = \tfrac{1}{4}(e^t + e^{-t})^2 = \tfrac{1}{4}(e^{2t} + 2 + e^{-2t})$$

and

$$\sinh^2 t = \tfrac{1}{4}(e^t - e^{-t})^2 = \tfrac{1}{4}(e^{2t} - 2 + e^{-2t}).$$

Subtracting gives formula (3).

Example 1 Show that $e^x = \cosh x + \sinh x$.

Solution By definition,

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Adding, $\cosh x + \sinh x = e^x/2 + e^{-x}/2 + e^x/2 - e^{-x}/2 = e^x$. ▲

Similarly, $e^{-x} = \cosh x - \sinh x$.

The other hyperbolic functions can be introduced by analogy with the trigonometric functions:

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x}, \tag{4}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{csch} x = \frac{1}{\sinh x}.$$

Various general identities can be proved exactly as we proved formula (3); for instance, the addition formulas are:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y, \tag{5a}$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y. \tag{5b}$$

Example 2 Prove identity (5a).

Solution By definition,

$$\sinh(x + y) = \frac{e^{x+y} - e^{-x-y}}{2} = \frac{e^x e^y - e^{-x} e^{-y}}{2}.$$

Now we plug in $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$ to get

$$\sinh(x + y) = \tfrac{1}{2}\big[(\cosh x + \sinh x)(\cosh y + \sinh y) - (\cosh x - \sinh x)(\cosh y - \sinh y)\big].$$

Expanding,

$$\begin{aligned}\sinh(x + y) &= \tfrac{1}{2}(\cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y + \sinh x \sinh y \\ &\qquad - \cosh x \cosh y + \cosh x \sinh y + \sinh x \cosh y - \sinh x \sinh y) \\ &= \cosh x \sinh y + \sinh x \cosh y. \ ▲\end{aligned}$$

Notice that in formula (5b) for $\cosh(x + y)$ there is no minus sign. This is one of several differences in signs between rules for the hyperbolic and circular functions. Another is in the following:

$$\frac{d}{dx} \sinh x = \cosh x, \tag{6a}$$

$$\frac{d}{dx} \cosh x = \sinh x. \tag{6b}$$

Example 3 Prove formula (6a).

Solution By definition, $\sinh x = (e^x - e^{-x})/2$, so

$$\frac{d}{dx} \sinh x = \frac{e^x - (-1)e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x. \ ▲$$

We also note that

$$\sinh(-x) = -\sinh x \qquad (\sinh \text{ is } \textit{odd})$$

and (7)

$$\cosh(-x) = \cosh x \qquad (\cosh \text{ is } \textit{even}).$$

From formulas (5a) and (5b) we get the half-angle formulas:

$$\sinh^2 x = \frac{\cosh 2x - 1}{2} \quad \text{and} \quad \cosh^2 x = \frac{\cosh 2x + 1}{2}. \tag{8}$$

Example 4 Prove that $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$.

Solution Since $\tanh x = \sinh x / \cosh x$, the quotient rule gives

$$(d/dx)\tanh x = (\cosh x \cdot \cosh x - \sinh x \cdot \sinh x)/\cosh^2 x = 1 - \tanh^2 x.$$

From $\cosh^2 x - \sinh^2 x = 1$ we get $1 - \sinh^2 x/\cosh^2 x = 1/\cosh^2 x$; that is, $1 - \tanh^2 x = \operatorname{sech}^2 x$. ▲

Hyperbolic Functions and Their Derivatives

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \frac{d}{dx}\sinh x = \cosh x,$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \frac{d}{dx}\cosh x = \sinh x,$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}, \qquad \frac{d}{dx}\tanh x = \operatorname{sech}^2 x,$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}, \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x,$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x}, \qquad \frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x,$$

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}} = \frac{1}{\sinh x}, \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x.$$

Example 5 Differentiate: (a) $\sinh(3x + x^3)$; (b) $\cos^{-1}(\tanh x)$; (c) $3x/(\cosh x + \sinh 3x)$.

Solution (a)

$$\frac{d}{dx}\sinh(3x + x^3) = \frac{d}{du}\sinh u \cdot \frac{du}{dx},$$

where $u = 3x + x^3$. We compute

$$\frac{d}{du}\sinh u = \cosh u \quad \text{and} \quad \frac{du}{dx} = 3 + 3x^2.$$

Expressing everything in terms of x, we have

$$\frac{d}{dx}\sinh(3x + x^3) = \cosh(3x + x^3) \cdot 3(1 + x^2).$$

(b) $$\frac{d}{dx}\cos^{-1}(\tanh x) = \frac{d}{du}\cos^{-1}u \cdot (du/dx), \qquad \text{where} \quad u = \tanh x.$$

We find

$$\frac{d}{du}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-\tanh^2 x}} = \frac{-1}{\operatorname{sech} x}$$

(from the identity $1-\tanh^2 x = \operatorname{sech}^2 x$ proved in Example 4), and

$$\frac{du}{dx} = \operatorname{sech}^2 x$$

(also from Example 4). Hence

$$\frac{d}{dx}\cos^{-1}(\tanh x) = -\operatorname{sech} x.$$

(c) $$\frac{d}{dx}\left(\frac{3x}{\cosh x + \sinh 3x}\right) = \frac{3(\cosh x + \sinh 3x) - 3x(\sinh x + 3\cosh 3x)}{(\cosh x + \sinh 3x)^2}$$

(by the quotient rule). ▲

Let us return now to the equation $d^2x/dt^2 - \omega^2 x = 0$. Its solution can be summarized as follows.

The Equation $d^2x/dt^2 - \omega^2 x = 0$

The solution of

$$\frac{d^2x}{dt^2} - \omega^2 x = 0 \tag{9}$$

is

$$x = x_0\cosh \omega t + \frac{v_0}{\omega}\sinh \omega t, \tag{10}$$

where $x = x_0$ and $dx/dt = v_0$ when $t = 0$.

That formula (10) gives a solution of equation (9) is easy to see:

$$\frac{dx}{dt} = \omega x_0 \sinh \omega t + v_0 \cosh \omega t,$$

using formula (6) and the chain rule. Differentiating again, we get

$$\frac{d^2x}{dt^2} = \omega^2 x_0 \cosh \omega t + \omega v_0 \sinh \omega t = \omega^2 x,$$

so equation (9) is verified.

One may prove that (10) gives the only solution of equation (9) just as in the case of the spring equation (Exercise 54).

Example 7 Solve for $f(t)$: $f'' - 3f = 0$, $f(0) = 1$, $f'(0) = -2$.

Solution We use formula (10) with $\omega^2 = 3$ (so $\omega = \sqrt{3}$), $x_0 = f(0) = 1$, $v_0 = f'(0) = -2$, and with $f(t)$ in place of x. Thus our solution is

$$f(t) = \cosh\sqrt{3}\,t - \frac{2}{\sqrt{3}}\sinh\sqrt{3}\,t. ▲$$

Example 8 Prove that $\cosh x$ has a minimum value of 1 at $x = 0$.

Solution $(d/dx)\cosh x = \sinh x$ vanishes only if $x = 0$ since $e^x = e^{-x}$ exactly when $e^{2x} = 1$; that is, $x = 0$. Also, $(d^2/dx^2)\cosh x = \cosh x$, which is 1 at $x = 0$, so $\cosh x$ has a minimum at $x = 0$, by the second derivative test. ▲

The kind of reasoning in the preceding example, together with the techniques of graphing, enable us to graph all the hyperbolic functions. These are shown in Fig. 8.3.3.

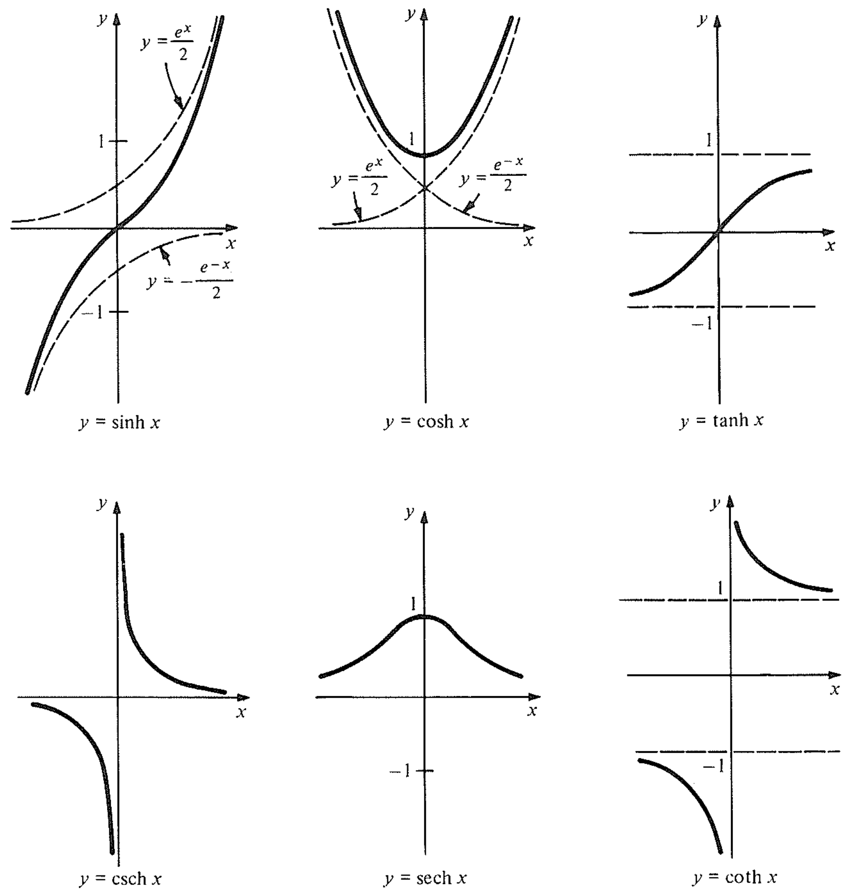


Figure 8.3.3. Graphs of the hyperbolic functions.

Finally, the differentiation formulas for the hyperbolic functions lead to integration formulas.

Antidifferentiation Formulas for Hyperbolic Functions

$$\int \cosh x\,dx = \sinh x + C, \qquad \int \operatorname{csch}^2 x\,dx = -\coth x + C,$$

$$\int \sinh x\,dx = \cosh x + C, \qquad \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C,$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C, \qquad \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C.$$

Example 9 Compute the integrals (a) $\int(\sinh 3x + x^3)\,dx$, (b) $\int \tanh x\,dx$, (c) $\int \cosh^2 x\,dx$, (d) $\int \frac{\sinh x}{1+\cosh^2 x}\,dx$.

Solution (a) $\int(\sinh 3x + x^3)\,dx = \int \sinh 3x\,dx + \int x^3\,dx = \frac{1}{3}\cosh 3x + \frac{x^4}{4} + C.$

(b) $\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \ln|\cosh x| + C = \ln(\cosh x) + C$

(since $\cosh x \geqslant 1$, $|\cosh x| = \cosh x$).

(c) Here we use the half-angle formula (8):

$$\int \cosh^2 x\,dx = \int \frac{\cosh 2x + 1}{2}\,dx = \frac{1}{4}\sinh 2x + \frac{x}{2} + C.$$

(d) $$\int \frac{\sinh x}{1+\cosh^2 x}\,dx = \int \frac{du}{1+u^2} \qquad (u = \cosh x)$$
$$= \tan^{-1}u + c = \tan^{-1}(\cosh x) + C. \ \blacktriangle$$

Supplement to Section 8.3: Unstable Equilibrium Points

In Section 8.1, we studied approximations to the differential equation of motion $m(d^2x/dt^2) = f(x)$, where the force function $f(x)$ satisfied the equilibrium condition $f(x_0) = 0$ at some position x_0. The linearized equation was

$$m\frac{d^2x}{dt^2} = f'(x_0)(x - x_0)$$

or, setting $y = x - x_0$, the displacement from equilibrium,

$$m\frac{d^2y}{dt^2} = f'(x_0)y.$$

If $f'(x_0) < 0$, this is the spring equation which has oscillating solutions given by trigonometric functions.

Now we can use the hyperbolic functions to study the case $f'(x_0) > 0$. The general solution is $y = A\cosh(\sqrt{f'(x_0)}\;t) + B\sinh(\sqrt{f'(x_0)}\;t)$, with A and B depending on the initial values of y and dy/dt, as in formula (10). We can also write this solution as $y = (A+B)e^{\sqrt{f'(x_0)}t} + (A-B)e^{-\sqrt{f'(x_0)}t}$. No matter how small the initial values, unless they are chosen so carefully that $A + B = 0$, the solution will approach $+\infty$ or $-\infty$ as $t \to \infty$. We say that the point x_0 is an *unstable equilibrium* position. In contrast to the stable case treated in Section 8.1, the linearization is not useful for all t, since most solutions eventually leave the region where the linear approximation is valid. Still, we can correctly conclude that solutions starting arbitrarily close to x_0 do not usually stay close, and that there are special solutions which approach x_0 as $t \to \infty$.[4]

[4] This analysis is useful in more advanced studies of differential equations. See, for instance, *Elementary Differential Equations and Boundary Value Problems*, by W. Boyce and R. DiPrima, Chapter 9, Third Edition, Wiley (1977) and *Differential Equations and Their Applications*, by M. Braun, Chapter 4, Third Edition, Springer-Verlag (1983).

For instance, let us find the unstable equilibrium point for the pendulum equation

$$m\frac{d^2\theta}{dt^2} = -m\frac{g}{l}\sin\theta = f(\theta).$$

First note that $f(\theta) = 0$ at $\theta = 0$ and π, corresponding to the bottom and top of pendulum swing (see Fig. 8.1.6). At $\theta = \pi$, $f'(\theta) = -(mg/l)\cos\theta = -(mg/l)\cdot(-1) = mg/l > 0$, so the top position is an unstable equilibrium point. At that point, the linearized equation is given by $(d^2/dt^2)(\theta - \theta_0) = (mg/l)(\theta - \theta_0)$, with solutions $\theta - \theta_0 = Ce^{\sqrt{mg/l}t} + De^{-\sqrt{mg/l}t}$. For most initial conditions, C will not equal zero, and the pendulum will move away from the equilibrium point. If the initial conditions are chosen just right, we will have $C = 0$, and the pendulum will gradually approach the top position as $t \to \infty$, but it will never arrive.

A very different application of hyperbolic functions, to the shape of a hanging cable, is given in Example 6, Section 8.5.

Exercises for Section 8.3

Prove the identities in Exercises 1–8.

1. $\tanh^2 x + \operatorname{sech}^2 x = 1.$
2. $\coth^2 x = 1 + \operatorname{csch}^2 x.$
3. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$
4. $\sinh^2 x = (\cosh 2x - 1)/2.$
5. $\frac{d}{dx}\cosh x = \sinh x.$
6. $\frac{d}{dx}\coth x = -\operatorname{csch}^2 x.$
7. $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x.$
8. $\frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x.$

Differentiate the functions in Exercises 9–24.

9. $\sinh(x^3 + x^2 + 2)$
10. $\tan^{-1}(\cosh x)$
11. $\sinh x \sinh 5x$
12. $\dfrac{\sinh x}{1 + \cosh x}$
13. $\sinh(\cos(8x))$
14. $\cos(\sinh(x^2))$
15. $\sinh^2 x + \cosh^2 x$
16. $\sinh^4 x + \cosh^4 x$
17. $\coth 3x$
18. $(\tanh x)(\operatorname{sech} x)$
19. $\exp(\tanh 2x)$
20. $\sin^{-1}(\tanh x)$
21. $\dfrac{\cosh x}{1 + \tanh x}$
22. $(\operatorname{csch} 2x)(1 + \tan x)$
23. $(\cosh x)\left(\int \dfrac{dx}{1 + \tanh^2 x}\right)$
24. $(\sinh x)\left(\int \dfrac{dx}{1 + \operatorname{sech}^2 x}\right)$

Solve the differential equations in Exercises 25–28.

25. $y'' - 9y = 0$, $y(0) = 0$, $y'(0) = 1$.
26. $f'' - 81f = 0$, $f(0) = 1$, $f'(0) = -1$.
27. $g'' - 3g = 0$, $g(0) = 2$, $g'(0) = 0$.
28. $h'' - 9h = 0$, $h(0) = 2$, $h'(0) = 4$.
29. Find the solution of the equation $d^2x/dt^2 - 9x = 0$, for which $x = 1$ and $dx/dt = 1$ at $t = 0$.
30. Solve $d^2y/dt^2 - 25y = 0$, where $y = 1$ and $dy/dt = -1$ when $t = 0$.
31. Find $f(t)$ if $f'' = 36f$, and $f(0) = 2$, $f'(0) = 0$.
32. Find $g(t)$ if $g''(t) = 25g(t)$, and $g(0) = 0$, $g'(0) = 1$.

Sketch the graphs of the functions in Exercises 33–36.

33. $y = 3 + \sinh x$
34. $y = (\cosh x) - 1$
35. $y = \tanh 3x$
36. $y = 3\cosh 2x$

Compute the integrals in Exercises 37–46.

37. $\int \cosh 3x\, dx$
38. $\int [\operatorname{csch}^2(2x) + (3/x)]\, dx$
39. $\int \coth x\, dx$
40. $\int x \tanh(x^2)\, dx$
41. $\int \sinh^2 x\, dx$
42. $\int \cosh^2 9y\, dy$
43. $\int e^x \sinh x\, dx$
44. $\int e^{2t}\cosh 2t\, dt$
45. $\int \cosh^2 x \sinh x\, dx$
46. $\int \dfrac{\sinh x}{\cosh^3 x}\, dx$

Compute dy/dx in Exercises 47–50.

47. $\dfrac{\sinh(x+y)}{xy} = 1$
48. $x + \cosh(xy) = 3$
49. $\tanh(3xy) + \sinh y = 1$
50. $\coth(x-y) - 3y = 6$

★51. (a) Find the *unstable* equilibrium point for the equation of motion $d^2x/dt^2 = x^2 - 1$.
(b) Write down the linearized equation of motion at this point.

★52. An atom of mass m in a linear molecule is subjected to forces of attraction by its neighbors given by

$$f(x) = -\frac{k_1}{(x-x_1)^2} + \frac{k_2}{(x-x_2)^2},$$

where $k_2 > k_1 > 0$, $0 < x_1 < x < x_2$. Show that $x_0 = (x_1 + \alpha x_2)/(1+\alpha)$, where $\alpha = \sqrt{k_1/k_2}$, is an unstable equilibrium. Write the linearized equations at this point.

★53. Prove the identity $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

★54. Prove that the equation $d^2x/dt^2 - \omega^2 x = 0$, with $x = x_0$ and $dx/dt = v_0$ when $t = 0$, has a unique solution given by $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$. [*Hint*: Study Exercises 31 and 32 of Section 8.1.] Why doesn't the energy method of p. 372 work in this case?

8.4 The Inverse Hyperbolic Functions

The inverse hyperbolic functions occur in several basic integration formulas.

We now study the inverses of the hyperbolic functions using the methods of Section 5.3. As with the inverse trigonometric functions, this yields some interesting integration formulas for algebraic functions.

We turn first to the inverse sinh function. Since $(d/dx)\sinh x = \cosh x$ is positive (Example 8, Section 8.3), $\sinh x$ is increasing. The range of $\sinh x$ is in fact $(-\infty, \infty)$ since $\sinh x \to \pm\infty$ as $x \to \pm\infty$. Thus, from the inverse function test in Section 5.3, we know that $y = \sinh x$ has an inverse function defined on the whole real line, denoted $\sinh^{-1} y$ by analogy with the notation for the inverse trigonometric functions. From the general formula

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)} \qquad (\text{where } y = f(x))$$

for the derivative of an inverse function, we get

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{(d/dx)(\sinh x)} = \frac{1}{\cosh x}.$$

From $\cosh^2 x - \sinh^2 x = 1$, we get

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{\cosh x} = \frac{1}{\sqrt{1+\sinh^2 x}} = \frac{1}{\sqrt{1+y^2}}. \tag{1}$$

The positive square root is taken because $\cosh x$ is always positive.

Example 1 Calculate (a) $\dfrac{d}{dx}\sinh^{-1}(3x)$ and (b) $\dfrac{d}{dx}[\sinh^{-1}(3\tanh 3x)]$.

Solution (a) Let $u = 3x$, so $\sinh^{-1} 3x = \sinh^{-1} u$. By the chain rule,

$$\frac{d}{dx}\sinh^{-1}(3x) = \left(\frac{d}{du}\sinh^{-1} u\right)\frac{du}{dx}.$$

By formula (1) with y replaced by u, we get

$$\frac{d}{dx}\sinh^{-1}(3x) = \frac{1}{\sqrt{1+u^2}}\,3 = \frac{3}{\sqrt{1+9x^2}}.$$

(b) By the formula $(d/dx)\sinh^{-1}x = 1/\sqrt{1+x^2}$, and the chain rule,

$$\begin{aligned}\frac{d}{dx}\sinh^{-1}(3\tanh 3x) &= \frac{1}{\sqrt{1+9\tanh^2 3x}}\cdot 3\frac{d}{dx}\tanh 3x\\ &= \frac{1}{\sqrt{1+9\tanh^2 3x}}\cdot 3\cdot 3\cdot \operatorname{sech}^2 3x\\ &= \frac{9\operatorname{sech}^2 3x}{\sqrt{1+9\tanh^2 3x}}. \blacktriangle\end{aligned}$$

There is an explicit formula for $\sinh^{-1}y$ obtained by solving the equation

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

for x. Multiplying through by $2e^x$ and gathering terms on the left-hand side of the equation gives $2e^x y - e^{2x} + 1 = 0$. Hence

$$(e^x)^2 - 2e^x y - 1 = 0,$$

and so, by the quadratic formula,

$$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}\,.$$

Since e^x is positive, we must select the positive square root. Thus, $e^x = y + \sqrt{y^2+1}$, and so $x = \sinh^{-1}y$ is given by

$$\sinh^{-1}y = \ln\left(y + \sqrt{y^2+1}\right). \tag{2}$$

The basic properties of $\sinh^{-1}$ are summarized in the following display.

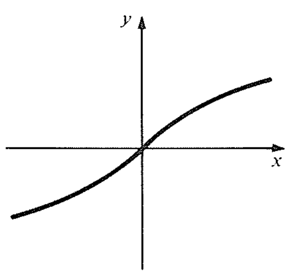


Figure 8.4.1. The graph of the $y = \sinh^{-1}x$.

Inverse Hyperbolic Sine Function

1. $\sinh^{-1}x$ is the inverse function of $\sinh x$; $\sinh^{-1}x$ is defined and is increasing for all x (Fig. 8.4.1); by definition: $\sinh^{-1}x = y$ is that number such that $\sinh y = x$.
2. $\dfrac{d}{dx}\sinh^{-1}x = \dfrac{1}{\sqrt{1+x^2}}$.
3. $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x + C = \ln(x+\sqrt{1+x^2}) + C.$

Example 2 Find $\sinh^{-1}5$ numerically by using logarithms.

Solution By (2), $\sinh^{-1}5 = \ln(5+\sqrt{5^2+1}) = \ln(5+\sqrt{26}) = \ln(10.100) \approx 2.31$. $\blacktriangle$

Example 3 Verify the formula $\displaystyle\int \frac{dx}{\sqrt{1+x^2}} = \ln(x+\sqrt{1+x^2}) + C.$

Solution

$$\begin{aligned}&\frac{d}{dx}\ln\left(x+\sqrt{1+x^2}\right)\\ &= \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left(1+\frac{x}{\sqrt{1+x^2}}\right) \quad \text{(by the chain rule)}\\ &= \left(\frac{1}{x+\sqrt{1+x^2}}\right)\left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right] = \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Thus the antiderivative for $1/\sqrt{1+x^2}$ is $\ln(x+\sqrt{1+x^2}) + C$. $\blacktriangle$

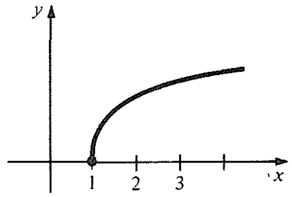


Figure 8.4.2. The graph of $y = \cosh^{-1}x$.

In a similar fashion we can investigate $\cosh^{-1}x$. Since $\cosh x$ is increasing on $[0, \infty)$ and has range $[1, \infty)$, $\cosh^{-1}x$ will be increasing, will be defined on $[1, \infty)$, and will have range $[0, \infty)$. Its graph can be obtained from that of $\cosh x$ by the usual method of looking through the page from the other side (Fig. 8.4.2).

By the same method that we obtained formula (1), we find

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}} \qquad (x > 1). \tag{3}$$

Example 4 Find $\dfrac{d}{dx}\cosh^{-1}(\sqrt{x^2+1}\,)$, $x \neq 0$.

Solution Let $u = \sqrt{x^2+1}$. Then, by the chain rule,

$$\begin{aligned}\frac{d}{dx}\cosh^{-1}\left(\sqrt{x^2+1}\,\right) &= \left(\frac{d}{du}\cosh^{-1}u\right)\cdot\left(\frac{du}{dx}\right)\\ &= \frac{1}{\sqrt{u^2-1}}\cdot\frac{x}{\sqrt{x^2+1}}\\ &= \frac{1}{\sqrt{x^2}}\cdot\frac{x}{\sqrt{x^2+1}} = \frac{x}{|x|}\cdot\frac{1}{\sqrt{x^2+1}}\,.\end{aligned}$$

Therefore,

$$\frac{d}{dx}\cosh^{-1}\left(\sqrt{x^2+1}\,\right) = \begin{cases}\dfrac{1}{\sqrt{x^2+1}} & \text{if} \quad x > 0,\\[2ex] \dfrac{-1}{\sqrt{x^2+1}} & \text{if} \quad x < 0. \ \blacktriangle\end{cases}$$

Similarly, we can consider $\tanh^{-1}x$ (see Fig. 8.4.3) and get, for $-1 < x < 1$,

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}\,. \tag{4}$$

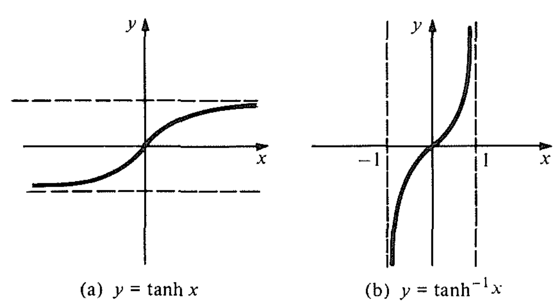


Figure 8.4.3. The graphs of $y = \tanh x$ and $y = \tanh^{-1}x$.

Example 5 Prove that $\tanh^{-1}x = \frac{1}{2}\ln[(1+x)/(1-x)]$, $-1 < x < 1$.

Solution Let $y = \tanh^{-1}x$, so

$$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{(e^y - e^{-y})}{(e^y + e^{-y})}\,.$$

Thus $x(e^y + e^{-y}) = e^y - e^{-y}$. Multiplying through by e^y and gathering terms on the left:

$$(x-1)e^{2y} + x + 1 = 0$$

$$e^{2y} = \frac{1+x}{1-x},$$

$$2y = \ln\left(\frac{1+x}{1-x}\right),$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

as required. ▲

Example 6 Show directly that the antiderivative of $\dfrac{1}{1-x^2}$ for $|x| < 1$ is $\dfrac{1}{2}\ln\dfrac{1+x}{1-x} + C$ by noticing that

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right).$$

Solution Since

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right),$$

an antiderivative is

$$-\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \qquad |x| \neq 1.$$

If $|x| < 1$, $(1+x)/(1-x) > 0$, so the absolute value signs can be removed. ▲

The remaining inverse functions are investigated in a similar way; the results are summarized in Fig. 8.4.4 and the box on the next page.

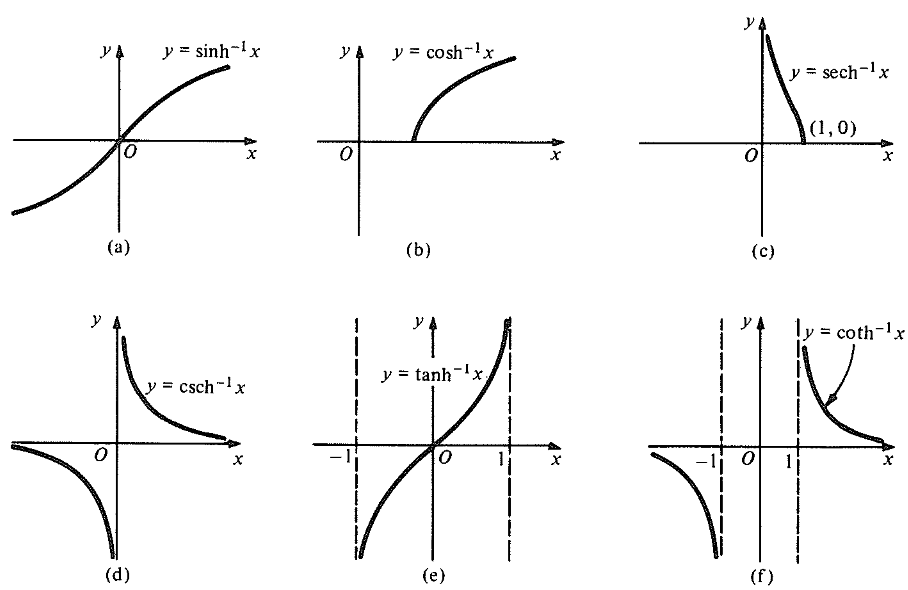


Figure 8.4.4. Graphs of the inverse hyperbolic functions.

The Inverse Hyperbolic Functions

$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$; $\qquad \dfrac{d}{dx}\sinh^{-1}x = \dfrac{1}{\sqrt{x^2+1}}$;

$\cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$; $\qquad \dfrac{d}{dx}\cosh^{-1}x = \dfrac{1}{\sqrt{x^2-1}}, \quad x > 1$;

$\tanh^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$; $\qquad \dfrac{d}{dx}\tanh^{-1}x = \dfrac{1}{1-x^2}, \quad |x| < 1$;

$\coth^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$; $\qquad \dfrac{d}{dx}\coth^{-1}x = \dfrac{1}{1-x^2}, \quad |x| > 1$;

$\operatorname{sech}^{-1}x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$; $\qquad \dfrac{d}{dx}\operatorname{sech}^{-1}x = \dfrac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$;

$$\operatorname{csch}^{-1}x = \begin{cases} \ln\left(\dfrac{1+\sqrt{1+x^2}}{x}\right), & x > 0; \\ -\ln\left(\dfrac{1+\sqrt{1+x^2}}{-x}\right), & x < 0; \end{cases} \qquad \frac{d}{dx}\operatorname{csch}^{-1}x = \begin{cases} \dfrac{-1}{x\sqrt{1+x^2}}, & x > 0, \\ \dfrac{1}{x\sqrt{1+x^2}}, & x < 0; \end{cases}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x + C = \ln\left(x + \sqrt{x^2+1}\right) + C;$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + C = \ln\left(x + \sqrt{x^2-1}\right) + C, \qquad x > 1;$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C, \qquad (|x| \neq 1),$$

$$= \begin{cases} \tanh^{-1}x + C = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) + C, & |x| < 1, \\ \coth^{-1}x + C = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right) + C, & |x| > 1; \end{cases}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1}x + C = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + C, \qquad 0 < x < 1;$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{csch}^{-1}x + C = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) + C, \qquad x > 0.$$

Example 7 Find $\displaystyle\int \frac{dx}{\sqrt{3x^2-1}}$.

Solution Going through the list of integration formulas for hyperbolic functions, we find $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + C$. Note that $\sqrt{3x^2-1} = \sqrt{(\sqrt{3}\,x)^2 - 1}$. Using the technique of integration by substitution with $u = \sqrt{3}\,x$ and $du = \sqrt{3}\,dx$,

we have

$$\int \frac{dx}{\sqrt{3x^2-1}} = \frac{1}{\sqrt{3}} \int \frac{du}{\sqrt{u^2-1}} = \frac{1}{\sqrt{3}} \cosh^{-1} u + C$$

$$= \frac{1}{\sqrt{3}} \cosh^{-1}(\sqrt{3}\,x) + C. \blacktriangle$$

Example 8 Find $\int \frac{dx}{\sqrt{4+x^2}}$.

Solution

$$\int \frac{dx}{\sqrt{x^2+4}} = \frac{1}{2} \int \frac{dx}{\left[\sqrt{1+(x/2)^2}\right]} = \frac{1}{2} \int \frac{2\,du}{\sqrt{1+u^2}} \qquad \left(u = \frac{x}{2}\right)$$

$$= \sinh^{-1} u + C = \sinh^{-1}\left(\frac{x}{2}\right) + C. \blacktriangle$$

Exercises for Section 8.4

Calculate the derivatives of the functions in Exercises 1–12.

1. $\cosh^{-1}(x^2+2)$
2. $\sinh^{-1}(x^3-2)$
3. $\sinh^{-1}(3x+\cos x)$
4. $\cosh^{-1}(x^2-\tan x)$
5. $x \tanh^{-1}(x^2-1)$
6. $x^2 \coth^{-1}(x+1)$
7. $\dfrac{x+\cosh^{-1}x}{\sinh^{-1}x+x}$
8. $\dfrac{1+\sinh^{-1}x}{1-\cosh^{-1}x}$
9. $\exp(1+\sinh^{-1}x)$
10. $\exp(3+\cosh^{-1}x)$
11. $\sinh^{-1}[\cos(3x)]$
12. $\cosh^{-1}[2+\sin(x^2)]$

In Exercises 13–16 calculate the indicated values numerically using logarithms.

13. $\tanh^{-1}(0.5)$
14. $\coth^{-1}(1.3)$
15. $\operatorname{sech}^{-1}(0.3)$
16. $\operatorname{csch}^{-1}(1.2)$

Derive the identities in Exercises 17–20.

17. $\cosh^{-1}x = \ln(x+\sqrt{x^2-1}), \quad |x|>1$
18. $\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \quad |x|>1$
19. $\operatorname{sech}^{-1}x = \ln\dfrac{1+\sqrt{1-x^2}}{x}, \quad 0<x<1$
20. $\operatorname{csch}^{-1}x = \ln\dfrac{1+\sqrt{1+x^2}}{x}, \quad x>0.$

Derive the identities in Exercises 21–24.

21. $\dfrac{d}{dx}\tanh^{-1}x = \dfrac{1}{1-x^2}.$
22. $\dfrac{d}{dx}\cosh^{-1}x = \dfrac{1}{\sqrt{x^2-1}}.$
23. $\dfrac{d}{dx}\operatorname{sech}^{-1}x = \dfrac{-1}{x\sqrt{1-x^2}}.$
24. $\dfrac{d}{dx}\coth^{-1}x = \dfrac{1}{1-x^2}.$

In Exercises 25–28 verify the given integration formulas by differentiation.

25. $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1}) + C, \ |x|>1.$
26. $\displaystyle\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + C, \ |x|>1.$
27. $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}} = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + C,$ $0<x<1.$
28. $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}} = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) + C, \ x>0.$

Calculate the integrals in Exercises 29–36.

29. $\displaystyle\int \frac{dx}{1-4x^2}$
30. $\displaystyle\int \frac{dx}{4x^2+1}$
31. $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$
32. $\displaystyle\int \frac{dx}{x\sqrt{1-4x^2}}$
33. $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x+1}}\,dx$
34. $\displaystyle\int \frac{e^x}{e^x\sqrt{1-e^{2x}}}\,dx$
35. $\displaystyle\int \frac{e^x}{1-e^{2x}}\,dx$
36. $\displaystyle\int \frac{\tan x\,dx}{\sqrt{1+\cos^2 x}}$

★37. Is the function $\cosh^{-1}(\sqrt{x^2+1})$ differentiable at all x?

8.5 Separable Differential Equations

Separable equations can be solved by separating the variables and integrating.

The previous sections dealt with detailed methods for solving particular types of differential equations, such as the spring equation and the equation of growth or decay. In this section and the next, we treat a few other classes of differential equations that can be solved explicitly, and we discuss a few general properties of differential equations.

A differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

in which the right-hand side factors into a product of a function of x and a function of y is called *separable*. Note that we use the term separable only for *first-order* equations; that is, equations involving only the first derivative of y with respect to x.

We may solve the above separable equation by rewriting it in differential notation[5] as

$$\frac{dy}{h(y)} = g(x)\,dx \qquad (\text{assuming } h(y) \neq 0)$$

and integrating:

$$\int \frac{dy}{h(y)} = \int g(x)\,dx.$$

If the integrations can be carried out, we obtain an expression relating x and y. If this expression can be solved for y, the problem is solved; otherwise, one has an equation that implicitly defines y in terms of x. The constant of integration may be determined by giving a value y_0 to y for a given value x_0 of x; that is, by specifying *initial conditions*.

Example 1 Solve $dy/dx = -3xy$, $y = 1$ when $x = 0$.

Solution We have

$$\frac{dy}{y} = -3x\,dx.$$

Integrating both sides gives

$$\ln|y| = -\frac{3x^2}{2} + C, \quad \text{and so} \quad y = \pm \exp C \exp\left(-\frac{3x^2}{2}\right).$$

Since $y = 1$ when $x = 0$, we choose the positive solution and $C = 0$, to give

$$y = \exp\left(-\frac{3x^2}{2}\right).$$

The reader may check by using the chain rule that this function satisfies the given differential equation. ▲

[5] For those worried about manipulations with differentials, answers obtained this way can always be checked by implicit differentiation.

The equation of growth (or decay) $y' = \gamma y$ studied in Section 8.2 is clearly separable, and the technique outlined above reproduces our solution $y = Ce^{\gamma x}$. The spring equation is *not* separable since it is of *second-order;* that is, it involves the second derivative of y with respect to x.

Separable Differential Equations

To solve the equation $y' = g(x)h(y)$:

1. Write
$$\frac{dy}{h(y)} = g(x)\,dx.$$
2. Integrate both sides:
$$\int \frac{dy}{h(y)} = \int g(x)\,dx + C.$$
3. Solve for y if possible.
4. The constant of integration C is determined by a given value of y at a given value of x, that is, by given *initial conditions.*

Example 2 Solve $dy/dx = y^2$, with $y = 1$ when $x = 1$, and sketch the solution.

Solution Separating variables and integrating, we get
$$\frac{dy}{y^2} = dx,$$
$$\frac{-1}{y} = x + C,$$
and so the general solution is
$$y = \frac{1}{-C - x}.$$
Substituting the initial conditions $y = 1$ and $x = 1$, we find that C must be -2, so the specific solution we seek in this case is
$$y = \frac{1}{2 - x}.$$

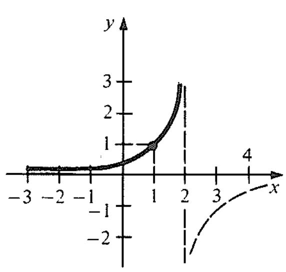


Figure 8.5.1. The solution of $y' = y^2, y(1) = 1$.

This function is sketched in Fig. 8.5.1. Notice that the graph has a vertical asymptote, and the function is undefined at $x = 2$. From the point of view of the differential equation, there is really nothing to justify using the portion of the function for $x > 2$, since the equation is not satisfied at $x = 2$. (One could imagine changing the value of C for $x > 2$ and obtaining a new function that still satisfies the differential equation and initial conditions.)

Thus we state that our solution is given by $y = 1/(2 - x)$ for x in $(-\infty, 2)$, and that the solution "blows up" at $x = 2$. ▲

Example 3 Solve
(a) $yy' = \cos 2x$, $y(0) = 1$;
(b) $dy/dx = x/(y + yx^2)$, $y(0) = -1$;
(c) $y' = x^2y^2 + x^2 - y^2 - 1$, $y(0) = 0$.

Solution (a) $y\,dy = \cos 2x\,dx$, so $y^2/2 = \frac{1}{2}\sin 2x + C$. Since $y = 1$ when $x = 0$, $C = \frac{1}{2}$. Thus $y^2 = \sin 2x + 1$ or $y = \sqrt{\sin 2x + 1}$. (We take the + square root since $y = +1$ when $x = 0$.)
(b) $y\,dy = x\,dx/(1 + x^2)$ so $y^2/2 = \frac{1}{2}\ln(1 + x^2) + C$, where the integration was done by substitution. Thus $y^2 = \ln(1 + x^2) + 2C$. Since $y = -1$ when $x = 0$, $C = \frac{1}{2}$. Since y is negative near $x = 0$, we choose the negative root: $y = -\sqrt{1 + \ln(1 + x^2)}$.
(c) The trick is to notice that the right-hand side factors: $y' = (x^2 - 1)(y^2 + 1)$. Thus $dy/(1 + y^2) = (x^2 - 1)\,dx$; integrating gives $\tan^{-1}y = (x^3/3) - x + C$. Since $y(0) = 0$, $C = 0$. Hence $y = \tan[(x^3/3) - x]$. ▲

Many interesting physical problems involve separable equations.

Example 4 **(Electric circuits)** We are told that the equation governing the electric circuit shown in Fig. 8.5.2 is

$$L\frac{dI}{dt} + RI = E$$

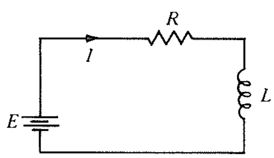


Figure 8.5.2. A simple electric circuit.

and that, in this case,

E (voltage) is a constant;
R (resistance) is a constant > 0;
L (inductance) is a constant > 0; and
I (current) is a function of time.

Solve this equation for I with a given value I_0 at $t = 0$.

Solution We separate variables:

$$L\frac{dI}{dt} = E - RI,$$

$$\frac{L}{E - RI}\,dI = dt$$

and then integrate:

$$-\frac{L}{R}\ln|E - RI| = t + C.$$

Thus

$$|E - RI| = \exp\left[-(t + C)\frac{R}{L}\right],$$

and so

$$E - RI = \pm\exp\left(-\frac{R}{L}t\right)\exp\left(-\frac{R}{L}C\right)$$

$$= A\exp\left(-\frac{Rt}{L}\right), \qquad \text{where} \quad A = \pm\exp\left(-\frac{R}{L}C\right).$$

At $t = 0$, $I = I_0$, so $E - RI_0 = A$. Substituting this in the previous equation and simplifying gives

$$I = \frac{E}{R} + \left(I_0 - \frac{E}{R}\right)e^{-Rt/L}.$$

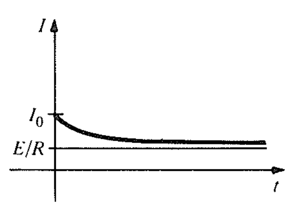


Figure 8.5.3. The current tends to the value E/R as $t \to \infty$.

As $t \to \infty$, I approaches the *steady state part* E/R, while $(I_0 - E/R)e^{-Rt/L}$, which approaches zero as $t \to \infty$, is called the *transient part of I*. (See Fig. 8.5.3.) ▲

Example 5 **(Predator-prey equations)** Consider x predators that feed on y prey. The numbers x and y change as t changes. Imagine the following model (called the *Lotka–Volterra model*).

(i) The prey increase by normal population growth (studied in Section 8.2), at a rate by (b is a positive birth rate constant), but decrease at a rate proportional to the number of predators and the number of prey, that is, $-rxy$ (r is a positive death rate constant). Thus

$$\frac{dy}{dt} = by - rxy.$$

(ii) The predators' population decreases at a rate proportional to their number due to natural decay (starvation) and increases at a rate proportional to the number of predators and the number of prey, that is,

$$\frac{dx}{dt} = -sx + cxy$$

for constants of starvation and consumption s and c.

If we eliminate t by writing

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \text{we get} \qquad \frac{dy}{dx} = \frac{by - rxy}{-sx + cxy}.$$

Solve this equation.

Solution The variables separate:

$$\frac{dy}{dx} = \frac{(b - rx)y}{x(-s + cy)},$$

$$\left(\frac{cy - s}{y}\right)dy = \left(\frac{b - rx}{x}\right)dx.$$

Integrating, we get

$$cy - s\ln y = b\ln x - rx + C$$

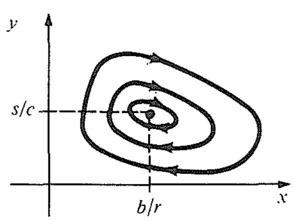


Figure 8.5.4. Solutions of the predator-prey equation.

for a constant C. This is an implicit form for the parametric curves followed by the predator–prey population. One can show that these curves are closed curves which surround the equilibrium point $(b/r, s/c)$ (the point at which $dx/dt = 0$ and $dy/dt = 0$), as shown in Fig. 8.5.4.[6]

Variants of this model are important in ecology for predicting and studying cyclic variations in populations. For example, this simple model already shows that if an insect prey and its predator are in equilibrium, killing both predators and prey with an insecticide can lead to a dramatic increase in the population of the prey, followed by an increase in the predators and so on, in cyclic fashion. Similar remarks hold for foxes and rabbits, etc. ▲

Example 6 **(The hanging cable)** Consider a freely hanging cable which weighs m kilograms per meter and is subject to a tension T_0. (See Fig. 8.5.5.) It can be shown[7] that the shape of the cable satisfies

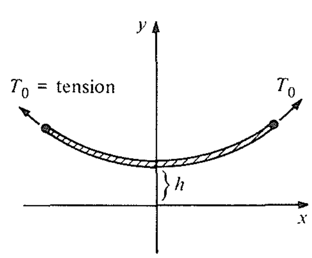


Figure 8.5.5. A cable hanging under its own weight.

$$\frac{d^2y}{dx^2} = \frac{mg}{T_0}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Introduce the new variable $w = dy/dx$ and show that w satisfies a separable equation. Solve for w and then y. You may assume the graph to be symmetric about the y axis.

[6] For a proof due to Volterra, see G. F. Simmons, *Differential Equations*, McGraw-Hill (1972) p. 286. There is also a good deal of information, including many references, in Chapter 9 of *Elementary Differential Equations and Boundary Value Problems* by W. Boyce and R. DiPrima, Third Edition, Wiley (1977), and in Section 1.5 of *Differential Equations and their Applications*, by M. Braun, Third Edition, Springer, (1983).

[7] See, for instance, T. M. Creese and R. M. Haralick, *Differential Equations for Engineers*, McGraw-Hill (1978) pp. 71–75.

Solution In terms of w, the equation for the cable is

$$\frac{dw}{dx} = \frac{mg}{T_0}\sqrt{1+w^2}\,.$$

Separating variables and integrating gives

$$\int \frac{dw}{\sqrt{1+w^2}} = \frac{mg}{T_0}\int dx,$$

$$\sinh^{-1}w = \frac{mg}{T_0}(x + C).$$

Since the cable is symmetric about the y axis, the slope $w = dy/dx$ is zero when $x = 0$, so the integration constant is zero.

Now $w = \sinh[(mg/T_0)x]$, and so

$$y = \int \frac{dy}{dx}\,dx = \int w\,dx = \int \sinh\left(\frac{mg}{T_0}x\right)dx = \frac{T_0}{mg}\cosh\left(\frac{mg}{T_0}x\right) + C_1\,.$$

The integration constant C_1 is found by setting $x = 0$. Since $\cosh(0) = 1$ and $y = h$ when $x = 0$ (Fig. 8.5.5), we get

$$h = T_0/mg + C_1, \quad \text{so} \quad C_1 = h - T_0/mg = (mgh - T_0)/mg.$$

Thus the equation for the shape of the cable is

$$y = \frac{T_0}{mg}\left[\cosh\left(\frac{mg}{T_0}x\right) - 1\right] + h.$$

The graph of $\cosh x$ takes its name *catenary* from this example and the Latin word *catena*, meaning "chain." ▲

We remark that cables which bear weight, such as the ones on suspension bridges, hang in a *parabolic* form (see Exercise 22).

Example 7 **(Orthogonal trajectories)** Consider the family of parabolas $y = kx^2$ for various constants k. (a) Find a differential equation satisfied by this family that does not involve k by differentiating and eliminating k. (b) Write a differential equation for a family of curves orthogonal (= perpendicular) to each of the parabolas $y = kx^2$ and solve it. Sketch.

Solution (a) If we differentiate, we have $y' = 2kx$; but $y = kx^2$ so $k = y/x^2$, and thus

$$y' = 2kx = 2\left(\frac{y}{x^2}\right)x = \frac{2y}{x}\,.$$

Thus any parabola $y = kx^2$ satisfies the equation $y' = 2y/x$.

(b) The slope of a line orthogonal to a line of slope m is $-1/m$, so the equation satisfied by the orthogonal trajectories is $y' = -x/2y$. This equation is separable:

$$2y\,dy = -x\,dx,$$

$$y^2 = -\frac{x^2}{2} + C,$$

$$y^2 + \frac{x^2}{2} = C.$$

If we write this as $y^2 + (x/\sqrt{2})^2 = C$, we see that these curves are obtained from the family of concentric circles with radii $\sqrt{C}$ centered at the origin by stretching the x axis by a factor of $\sqrt{2}$. (See Fig. 8.5.6.) They are *ellipses*. (See Section 14.1 for a further discussion of ellipses.) ▲

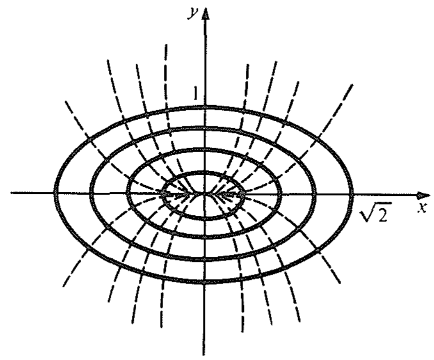


Figure 8.5.6. The orthogonal trajectories of the family of parabolas are ellipses.

Separable differential equations are a special case of the equation

$$\frac{dy}{dx} = F(x, y),$$

where F is a function depending on both x and y.[8] For example,

$$\frac{dy}{dx} = -x^2y + y^3 + 3\sin y + 1$$

is a differential equation that is not separable. There is little hope of solving such equations explicitly, except in rather special cases, such as the separable case. In general, one has to resort to numerical or other approximate methods. To do so, it is useful to have a geometric picture of what is going on.

The given data $dy/dx = F(x, y)$ tell us the slope at each point of the solution $y = f(x)$ that we seek. We can therefore imagine drawing small lines in the xy plane, with slope $F(x, y)$ at the point (x, y), as in Fig. 8.5.7.

The problem of finding a solution to the differential equation is precisely the problem of threading our way through this *direction field* with a curve which is tangent to the given direction at each point. (See Fig. 8.5.8. In this figure, some of the line segments are vertical, reflecting the fact that the formula for $F(x, y)$ may be a fraction whose denominator is sometimes zero.)

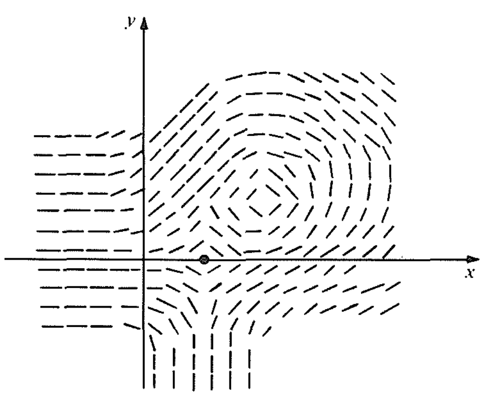


Figure 8.5.7. A plot of a direction field.

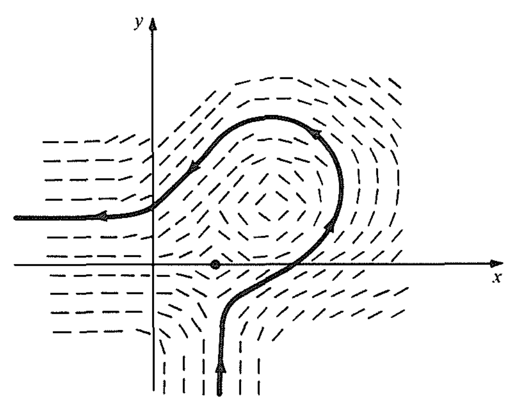


Figure 8.5.8. A solution threads its way through the direction field.

[8] We study such functions in detail beginning with Chapter 14. The material of those later chapters is not needed here.

We saw in Example 5 that differential equations may be given in parametric form

$$\frac{dx}{dt} = g(x, y), \quad \text{and} \quad \frac{dy}{dt} = h(x, y).$$

(so $dy/dx = h(x, y)/g(x, y)$.) Here we seek a parametric curve $(x(t), y(t))$ solving these two equations. From our discussion of parametric curves in Section 2.4, we see that the pair $(g(x, y), h(x, y))$ gives the velocity of the solution curve passing through (x, y). In this formulation, we can interpret Fig. 8.5.7 as a *velocity field*. If one thinks of the motion of a fluid, one can phrase the problem of finding solutions to the above pair of differential equations as follows: given the velocity field of a fluid, find the paths that fluid particles follow. For this reason, a solution curve is often called a *flow line*.

Example 8 Sketch the direction field for the equation $dy/dx = -x/y$ and solve the equation.

Solution Here the slope at (x, y) is $-x/y$. We draw small line segments with these slopes at a number of selected locations to produce Fig. 8.5.9.

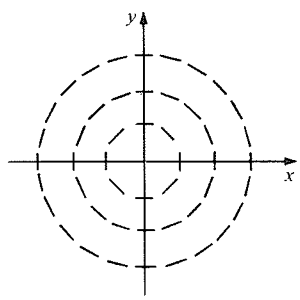


Figure 8.5.9. The direction field for $y' = -x/y$.

The equation is separable:

$$y\,dy = -x\,dx,$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C,$$

$$y^2 + x^2 = 2C.$$

Thus any solution is a circle and the solutions taken together form a family of circles. This is consistent with the direction field. ▲

When a numerical technique is called for, the direction field idea suggests a simple method. This procedure, called the *Euler method*, replaces the actual solution curve by a polygonal line and follows the direction field by moving a short distance along a straight line. For $dy/dx = F(x, y)$ we start at (x_0, y_0) and break up the interval $[x_0, x_0 + a]$ into n steps $x_0, x_1 = x_0 + a/n$, $x_2 = x_0 + 2a/n, \ldots, x_n = x_0 + a$. Now we recursively define

$$y_1 = F(x_0, y_0)\frac{a}{n} + y_0$$

$$y_2 = F(x_1, y_1)\frac{a}{n} + y_1$$

$$\vdots \qquad \vdots$$

$$y_n = F(x_{n-1}, y_{n-1})\frac{a}{n} + y_{n-1};$$

that is,

$$y_i - y_{i-1} = \left[\frac{dy}{dx}(x_{i-1}, y_{i-1}) \right](x_i - x_{i-1}), \qquad i = 1, 2, \dots, n$$

to produce the desired approximate solution (the polygonal curve shown in Fig. 8.5.10).

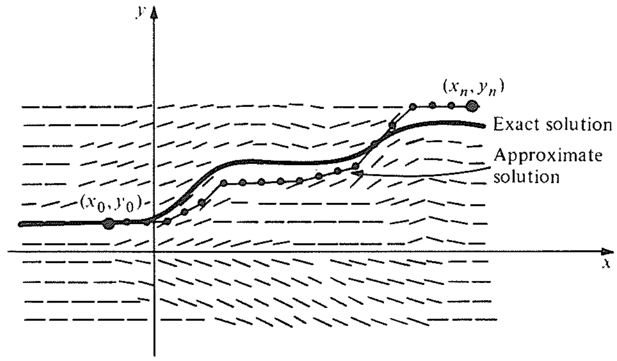
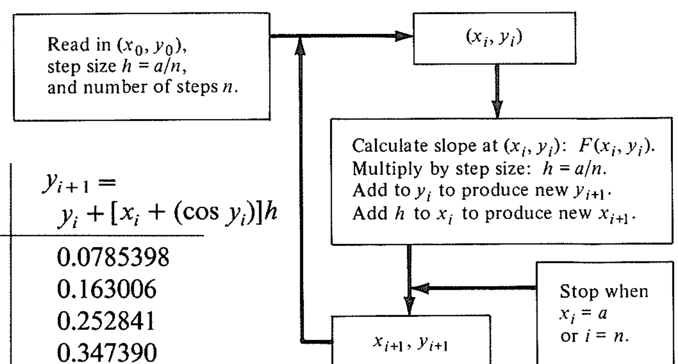


Figure 8.5.10. The Euler method for numerically solving differential equations.

Example 9 Solve the equation $dy/dx = x + \cos y$, $y(0) = 0$ from $x = 0$ to $x = \pi/4$ using a ten-step Euler method; that is, find $y(\pi/4)$ approximately.

Solution The recursive procedure is summarized below. It is helpful to record the data in a table as you proceed.[9] Here $x_0 = 0$, $y_0 = 0$, $a = \pi/4$, $n = 10$; thus $h = a/n = \pi/40 = 0.0785398$.



	x_i	y_i	$x_{i+1} = x_i + h$	$y_{i+1} = y_i + [x_i + (\cos y_i)]h$
$i = 0$	0 START	0	0.0785398	0.0785398
$i = 1$	0.0785398	0.0785398	0.157080	0.163006
$i = 2$	0.157080	0.163006	0.235619	0.252841
$i = 3$	0.235619	0.252841	0.314159	0.347390
$i = 4$	0.314159	0.347390	0.392699	0.445912
$i = 5$	0.392699	0.445912	0.471239	0.547614
$i = 6$	0.471239	0.547614	0.549779	0.651680
$i = 7$	0.549779	0.651680	0.628318	0.757304
$i = 8$	0.628319	0.757304	0.706858	0.863726
$i = 9$	0.706858	0.863726	0.785398	0.970263
$i = 10$	0.785398	0.970263	STOP	

Thus $y(\pi/4) \approx 0.970263$. ▲

[9] The Euler or related methods are particularly easy to use with a programmable calculator. In practice, the Euler method is not the most accurate or efficient. Usually the Runge–Kutta or predictor–corrector method is more accurate. (For details and comparative error analyses, a book such as C. W. Gear, *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall Englewood Cliffs, N.J. (1971) should be consulted.)

Direction fields can also be used to derive some qualitative information without solving the differential equation.

Example 10 Sketch the direction field for the equation $dy/dx = y^2 - 3y + 2$ and use it to find $\lim_{x\to\infty} y(x)$ geometrically for a solution satisfying $1 < y(0) < 2$.

Solution First we factor $y^2 - 3y + 2 = (y-1)(y-2)$. Thus $y^2 - 3y + 2$ is negative on the interval $(1, 2)$ and positive on the intervals $(2, \infty)$ and $(-\infty, 1)$. The direction field in the xy plane, which is independent of x, can now be plotted, as in Fig. 8.5.11.

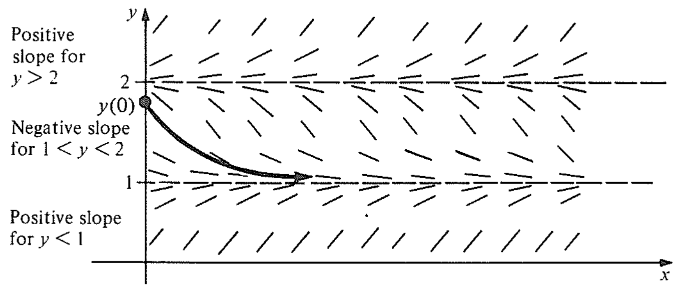


Figure 8.5.11. Direction field for the equation $dy/dx = y^2 - 3y + 2$.

If $1 < y(0) < 2$, then $y(x)$ must thread its way through the direction field, always remaining tangent to it. From Fig. 8.5.11, it is clear that a solution with initial condition $y(0)$ lying in the interval $(1, 2)$ is pushed downward, flattens out and becomes asymptotic to the line $y = 1$. Thus $\lim_{x\to\infty} y(x) = 1$. ▲

Supplementary Remark. Figure 8.5.11 enables us to see that the *equilibrium* solution $y = 1$ is stable; in other words, any initial condition $y(0)$ close to 1 gives a solution which remains close to 1 for all x. Compare this with the discussion of stable oscillations in the Supplement to Section 8.1. Likewise, the solution $y = 2$ is unstable. Again, consult the discussion of unstable equilibria in the Supplement to Section 8.3.

Exercises for Section 8.5

In Exercises 1–12 find the solution of the given differential equation satisfying the stated conditions (express your answer implicitly if necessary).

1. $\dfrac{dy}{dx} = \cos x$, $y(0) = 1$.
2. $\dfrac{dy}{dx} = y\cos x$, $y(0) = 1$.
3. $\dfrac{dy}{dx} = 2xy - 2y + 2x - 2$, $y(1) = 0$.
4. $y\dfrac{dy}{dx} = x$, $y(0) = 1$.
5. $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}$, $y(1) = -2$
6. $x\dfrac{dy}{dx} = \sqrt{1-y^2}$, $y(1) = 0$.
7. $\dfrac{dy}{dx} = \dfrac{xe^{-y}}{(x^2+1)y}$, $y(0) = 1$.
8. $\dfrac{dy}{dx} = y^3\dfrac{\sin x}{1+8y^4}$, $y(0) = 1$.
9. $\dfrac{dy}{dx} = \dfrac{1+y}{1+x}$, $y(0) = 1$.
10. $\dfrac{dy}{dx} = 3xy - x$.
11. $\dfrac{dy}{dx} = \cos x - y\cos x$, $y(0) = 2$.
12. $e^y\left(\dfrac{dy}{dx}\right) = 1 + e^{2y} - xe^{2y} - x$, $y(0) = 1$.
13. The current I in an electric circuit is described by the equation $3(dI/dt) + 8I = 10$, and the initial current is $I_0 = 2.1$ at $t = 0$. Sketch the graph of I as a function of time.
14. Repeat Exercise 13 for $I_0 = 0.3$.
15. *Capacitor equation*. The equation $R(dQ/dt) + Q/C = E$ describes the charge Q on a capacitor, where R, C, and E are constants. (See Fig. 8.5.12.) (a) Find Q as a function of time if $Q = 0$ at $t = 0$. (b) How long does it take for Q to attain 99% of its limiting charge?

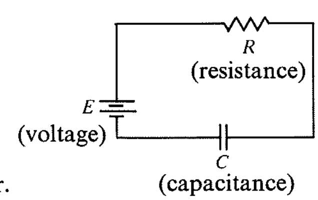


Figure 8.5.12. A circuit with a charging capacitor.

16. Repeat Exercise 15 for $Q = 3.1$ at $t = 0$ and $R = 2$, $E = 10$, and $C = 2$.
17. Verify directly that $x = b/r$, $y = s/c$ is a solution of the predator–prey equations (see Example 5).
18. (a) Verify graphically that the equation $y - \ln y = c$ has exactly two positive solutions if $c > 1$. (b) What does this have to do with Fig. 8.5.4?
19. *Logistic law of population growth*. If a population can support only P_0 members, the rate of growth of the population may be given by $dP/dt = kP(P_0 - P)$. This modification of the law of growth $dP/dt = \alpha P$ discussed in Section 8.2 is called the logistic law, or the Verhulst law. Solve this equation and show that P tends to P_0 as $t \to \infty$. *Hint*: In solving the equation you may wish to use the identity
$$\frac{1}{P(P_0 - P)} = \frac{1}{P_0}\left(\frac{1}{P} + \frac{1}{P_0 - P}\right).$$
20. *Chemical reaction rates*. Chemical reactions often proceed at a rate proportional to the concentrations of each reagent. For example, consider a reaction of the type $2A + B \to C$ in which two molecules of A and one of B combine to produce one molecule of C. Concentrations are measured in moles per liter, where a mole is a definite number (6×10^{23}) of molecules. Let the concentrations of A, B, and C at time t be a, b, c, and suppose that $c = 0$ at $t = 0$. Since no molecules are destroyed, $b_0 - b = (a_0 - a)/2$, where a_0 and b_0 are the values of a and b at $t = 0$. The rate of change of a is given by $da/dt = ka^2b$ for a constant k. Solve this equation. [*Hint*: You will need to make up an identity like the one used in Exercise 19.]
21. In Example 6, verify that the solution becomes straight as T_0 increases to ∞.
22. *Suspension bridge*. The function $y(x)$ describing a suspended cable which weighs m kilograms per meter and which is subject to a tension T_0 satisfies $dy/dx = (m/T_0)x$. Verify that the cable hangs in a parabolic shape.
23. (a) Find a differential equation satisfied by the family of hyperbolas $xy = k$ for various constants k.
(b) Find a differential equation satisfied by the orthogonal trajectories to the hyperbolas $xy = k$, solve it, and sketch the resulting family of curves.
24. (a) Find a differential equation satisfied by the family of ellipses, $x^2 + 4y^2 = k$, k a constant.
(b) Find a differential equation satisfied by the orthogonal trajectories to this family of ellipses, solve it and sketch.
25. (a) Sketch the family of cubics $y = cx^3$, when c is constant.
(b) Find a differential equation satisfied by this family.
(c) Find a differential equation for the orthogonal family and solve it.
26. Repeat Exercise 25 for the family of cubics $y = x^3 - dx$, where d is constant.
27. (a) Sketch the direction field for the equation $dy/dx = 2y/x$. (b) Solve this equation.
28. Sketch the direction field for the equation $y' = -y/x$. Solve the equation and show that the solutions are consistent with your direction field.
29. Use a ten-step Euler method to find y approximately at $x = 1$ if $dy/dx = y - x^2$ and $y(0) = 1$.
30. If $y' = x + \tan y$, $y(0) = 0$, find $y(1)$ approximately using a ten-step Euler procedure.
31. Find an approximate solution for $y(1)$ if $y' = x\sqrt{1 + y^4}$ and $y(0) = 0$ using a fifteen-step Euler method.
32. Redo Example 9 using a twenty-step Euler method, compare the answers and discuss.

In Exercises 33–36, sketch the direction field of the given equation and use it to find $\lim_{x\to\infty} y(x)$ geometrically for the given $y(0)$.

33. $\dfrac{dy}{dx} = y^2 - 7y + 12$, $y(0) = 3.5$.
34. $\dfrac{dy}{dx} = y^2 + y - 2$, $y(0) = -2$.
35. $\dfrac{dy}{dx} = y^3 - 6y^2 + 11y - 6$, $y(0) = 1.5$.
36. $\dfrac{dy}{dx} = y^3 - 4y^2 + 3y$, $y(0) = 2$.
37. Suppose that $y = f(x)$ solves the equation $dy/dx = e^xy^2 + 4xy^5$, $f(0) = 1$. Calculate $f'''(0)$.
38. Suppose $y'' + 3(y')^3 + 8e^xy^2 = 5\cos x$, $y'(0) = 1$, and $y''(0) = 2$. Calculate $y'''(0)$.
★39. Consider a family of curves defined by a separable equation $dy/dx = g(x)h(y)$. Express the family of orthogonal trajectories implicitly in terms of integrals.
★40. Show by a graphical argument that any straight line through $(b/r, s/c)$ meets the curve $cy - s\ln y = b\ln x - rx + C$ (for $C > 0$) at exactly two points. (See Example 5).

8.6 Linear First-Order Equations

First-order equations which are linear in the unknown function can be solved explicitly.

We have seen that separable equations can be solved directly by integration. There are a few other classes of differential equations that can be solved by reducing them to integration after a suitable transformation. We shall treat one such class now. (Other classes are discussed in Sections 12.7 and 18.3.)

We consider equations that are linear in the unknown function y:

$$\frac{dy}{dx} = P(x)y + Q(x) \tag{1}$$

for given functions P and Q of x. If Q is absent, equation (1) becomes

$$\frac{dy}{dx} = P(x)y \tag{2}$$

which is separable:

$$\frac{1}{y}\,dy = P(x)\,dx,$$

$$\ln|y| = \int P(x)\,dx + C,$$

$$|y| = \exp(C)\exp\left(\int P(x)\,dx\right).$$

Choosing $C = 0$ and $y > 0$ gives the particular solution

$$y = \exp\left(\int P(x)\,dx\right). \tag{3}$$

Now we use the solution (3) of equation (2) to help us simplify equation (1). If y solves equation (1), we divide it by the function (3), obtaining a new function

$$w = y\exp\left(-\int P(x)\,dx\right) \tag{4}$$

which turns out to satisfy a *simpler* equation. By the product and chain rules, we get

$$\begin{aligned}\frac{dw}{dx} &= \frac{dy}{dx}\exp\left(-\int P(x)\,dx\right) - yP(x)\exp\left(\int -P(x)\,dx\right)\\ &= \left[P(x)y + Q(x)\right]\exp\left(-\int P(x)\,dx\right) - P(x)\,y\exp\left(\int -P(x)\,dx\right).\end{aligned}$$

The terms involving y cancel, leaving

$$\frac{dw}{dx} = Q(x)\exp\left(-\int P(x)\,dx\right).$$

The right-hand side is a function of x alone, so we may integrate:

$$w = \int Q(x)\left[\exp\left(-\int P(x)\,dx\right)\right]dx + C. \tag{5}$$

Combining formulas (4) and (5) gives the general solution y of equation (1), written out explicitly in the following box.

Linear First-Order Equations

The general solution of

$$\frac{dy}{dx} = P(x)y + Q(x)$$

is

$$y = \exp\left(\int P(x)\,dx\right)\left\{\int Q(x)\left[\exp\left(-\int P(x)\,dx\right)\right]dx + C\right\}, \qquad (6)$$

where C is an arbitrary constant.

One may verify by direct substitution that the expression (6) for y in this display solves equation (1). Instead of memorizing *formula* (6) for the solution, it may be easier to remember the *method*, as summarized in the following box.

Method for Solving $dy/dx = P(x)y + Q(x)$

1. Calculate $\int P(x)\,dx$, dropping the integration constant.
2. Transpose $P(x)y$ to the left side: $\dfrac{dy}{dx} - P(x)y = Q(x)$.
3. Multiply the equation by $\exp(-\int P(x)\,dx)$.
4. The left-side of the equation should now be a derivative:
   $$\frac{d}{dx}\left[y\exp\left(-\int P(x)\,dx\right)\right].$$
   Check to make sure.
5. Integrate both sides, keeping the constant of integration.
6. Solve the resulting equation for y.
7. Use the initial condition, if given, to solve for the integration constant.

The expression $\exp(-\int P(x)\,dx)$ is called an *integrating factor*, since multiplication by this term enables us to solve the equation by integration.

Example 1 Solve $dy/dx = xy + x$.

Solution We follow the procedure in the preceding box.

1. $\quad P(x) = x, \quad$ so $\quad \int P(x)\,dx = \dfrac{x^2}{2}$.

2. $$\frac{dy}{dx} - xy = x$$

(transposing xy to the left-hand side).

3. $$\exp\left(-\frac{x^2}{2}\right)\left\{\frac{dy}{dx} - xy\right\} = \exp\left(-\frac{x^2}{2}\right)x$$

(multiplying by $\exp(-x^2/2)$).

4. $$\frac{d}{dx}\left\{y\exp\left(-\frac{x^2}{2}\right)\right\} = \frac{dy}{dx}\exp\left(-\frac{x^2}{2}\right) - yx\exp\left(-\frac{x^2}{2}\right)$$

which equals the left-hand side in step 3. Thus the equation is now

$$\frac{d}{x}\left\{y\exp\left(-\frac{x^2}{2}\right)\right\} = x\exp\left(-\frac{x^2}{2}\right).$$

5. Integration, using the substitution $u = -x^2/2$, yields

$$y\exp\left(-\frac{x^2}{2}\right) = \int x\exp\left(-\frac{x^2}{2}\right)dx = -\exp\left(-\frac{x^2}{2}\right) + C.$$

6. Solving for y,

$$y = C\exp\left(\frac{x^2}{2}\right) - 1. \blacktriangle$$

Example 2 Solve the following equations with the stated initial conditions.

(a) $y' = e^{-x} - y$, $y(0) = 1$;
(b) $y' = \cos^2 x - (\tan x)y$, $y(0) = 1$.

Solution (a) The integrating factor is $\exp(-\int P(x)\,dx) = \exp(-\int -dx) = \exp(x)$. Thus

$$e^x(y' + y) = 1,$$

$$\frac{d}{dx}(e^x y) = 1,$$

$$e^x y = x + C.$$

Since $y(0) = 1$, $C = 1$, so $y = (x+1)e^{-x}$.
(b) The integrating factor is $\exp(\int \tan x\,dx) = \exp(-\ln\cos x) = 1/\cos x$. (This is valid only if $\cos x > 0$, but since our initial condition is $x = 0$ where $\cos x = 1$, this is justified.) Thus

$$\frac{1}{\cos x}\left[y' + (\tan x)y\right] = \cos x,$$

$$\frac{d}{dx}\left[\frac{y}{\cos x}\right] = \cos x,$$

$$\frac{y}{\cos x} = \sin x + C.$$

Since $y = 1$ when $x = 0$, $C = 1$. Thus $y = \cos x\sin x + \cos x$. It may be verified that this solution is valid for all x. ▲

Example 3 **(Electric circuits)** In Example 4, Section 8.5, replace E by the sinusoidal voltage

$$E = E_0\sin\omega t$$

with L, R, and E_0 constants, and solve the resulting equation.

Solution The equation is

$$\frac{dI}{dt} = -\frac{RI}{L} + \frac{E_0}{L}\sin\omega t.$$

We follow the procedure in the preceding box with x replaced by t and y by I:

1. $P(t) = -\dfrac{R}{L}$, a constant, so $\int P(t)\,dt = -tR/L$.

2. $\dfrac{dI}{dt} + \dfrac{RI}{L} = \dfrac{E_0}{L}\sin\omega t.$

3. $\left[\exp\left(\dfrac{tR}{L}\right)\right]\left(\dfrac{dI}{dt} + \dfrac{RI}{L}\right) = \dfrac{E_0}{L}\exp\left(\dfrac{tR}{L}\right)\sin\omega t.$

4. $\dfrac{d}{dt}\left\{\exp\left(\dfrac{tR}{L}\right)I\right\} = \dfrac{E_0}{L}\exp\left(\dfrac{tR}{L}\right)\sin\omega t.$

5. $\exp\left(\dfrac{tR}{L}\right)I = \dfrac{E_0}{L}\displaystyle\int \exp\left(\dfrac{tR}{L}\right)\sin\omega t\, dt.$

This integral may be evaluated by the method of Example 4, Section 7.4, namely integration by parts twice. One gets

$$\exp\left(\frac{tR}{L}\right)I = \frac{E_0}{L}\left\{\frac{e^{tR/L}}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right)\right\} + C.$$

6. Solving for I,

$$I = \frac{E_0}{L}\,\frac{1}{(R/L)^2+\omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + Ce^{-tR/L}.$$

The constant C is determined by the value of I at $t=0$. This expression for I contains an oscillatory part, oscillating with the same frequency ω as the driving voltage (but with a phase shift; see Exercise 10) and a transient part $Ce^{-tR/L}$ which decays to zero as $t\to\infty$. (See Fig. 8.6.1.) ▲

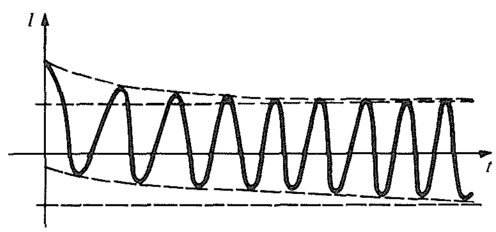


Figure 8.6.1. The graph of the solution of a sinusoidally forced electric circuit containing a resistor and an inductor.

Example 4 **(Pollution)** A small lake contains 4×10^7 liters of pure water at $t=0$. A polluted stream carries 0.67 liter of pollutant and 10 liters of water into the lake per second. (Assume that this mixes instantly with the lake water.) Meanwhile, 10.67 liters per second of the lake flow out in a drainage stream. Find the amount of pollutant in the lake as a function of time. What is the limiting value?

Solution Let $y(t)$ denote the amount of pollutant in liters in the lake at time t. The amount of pollutant in one liter of lake water is thus

$$\frac{y(t)}{4\times10^7}.$$

The rate of change of $y(t)$ is the rate at which pollutant flows out, which is $-10.67y(t)/4\times10^7 = -2.67\times10^{-7}y(t)$ liters per second, plus the rate at which it flows in, which is 0.67 liter per second. Thus

$$y' = -2.67\times10^{-7}y + 0.67.$$

The solution of $dy/dt = ay+b$, $y(0)=0$, is found using the integrating factor e^{-at}:

$$e^{-at}(y'-ay) = be^{-at},$$

$$\frac{d}{dt}(e^{-at}y) = be^{-at},$$

$$e^{-at}y = -\frac{b}{a}e^{-at} + C.$$

Since $y = 0$ at $t = 0$, $C = b/a$. Thus

$$e^{-at}y = \frac{b}{a}(1 - e^{-at}),$$

$$y = \frac{b}{a}(e^{at} - 1).$$

Here $a = -2.67 \times 10^{-7}$ and $b = 0.67$, so $y = (1 - e^{-2.67\times10^{-7}t})(2.51 \times 10^6)$ liters. For t small, y is relatively small; but for larger t, y approaches the (steady-state) catastrophic value of 2.51×10^6 liters, that is, the lake is well over half pollutant. (See Exercise 13 to find out how long this takes.) ▲

Example 5 **(Falling object in a resisting medium)** The downward force acting on a body of mass m falling in air is mg, where g is the gravitational constant. The force of air resistance is γv, where γ is constant of proportionality and v is the downward speed. If a body is released from rest, find its speed as a function of time t. (Assume it is released from a great enough height so that it has not hit the ground by time t.)

Solution Since mass times acceleration is force, and acceleration is the time derivative of velocity, we have the equation $m(dv/dt) = mg - \gamma v$ or, equivalently, $dv/dt = -(\gamma/m)v + g$ which is a linear first-order equation. Its general solution is

$$v = e^{-\gamma t/m}\int g e^{\gamma t/m}\,dt = e^{-\gamma t/m}\left(\frac{mg}{\gamma}e^{\gamma t/m} + C\right).$$

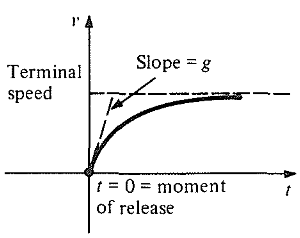


Figure 8.6.2. The speed of an object moving in a resisting medium.

If $v = 0$ when $t = 0$, C must be $-mg/\gamma$, so $v = (mg/\gamma)(1 - e^{-\gamma t/m})$. Note that as $t \to \infty$, $e^{-\gamma t/m} \to 0$ and so $v \to mg/\gamma$ the *terminal* speed. (See Fig. 8.6.2.) For small t the velocity is approximately gt, which is what it would be if there were no air resistance. As t increases, the air resistance slows the velocity and a terminal velocity is approached. ▲

Example 6 **(Rocket propulsion)** A rocket with an initial mass M_0 (kilograms) blasts off at time $t = 0$ (Fig. 8.6.3). The mass decreases with time because the fuel is being

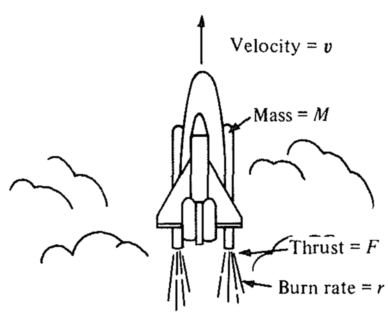


Figure 8.6.3. A rocket blasting off at $t = 0$.

spent at a constant burn rate r (kilograms per second). Thus, the mass at time t is $M = M_0 - rt$. If the thrust is a constant force F, and the velocity is v, Newton's second law gives

$$\frac{d}{dt}(Mv) = F - Mg, \tag{7}$$

where $g = 9.8$ is the gravitational constant. (We neglect air resistance and assume the motion to be vertical.)

(a) Solve equation (7).
(b) If the mass of the rocket at burnout is M_1, compute the velocity at burnout.

Solution (a) Substituting $M = M_0 - rt$ into equation (7) gives

$$\frac{d}{dt}\left[(M_0 - rt)v\right] = F - (M_0 - rt)g.$$

Although this is a linear equation in v, it is already in a form which we can directly integrate:

$$(M_0 - rt)v = \int (F - g(M_0 - rt))\,dt + C$$

$$= Ft - g\left(M_0 t - \frac{rt^2}{2}\right) + C.$$

Solving for v,

$$v = \frac{Ft}{M_0 - rt} - \frac{g}{M_0 - rt}\left(M_0 t - \frac{rt^2}{2}\right) + \frac{C}{(M_0 - rt)}.$$

Since $v = 0$ at $t = 0$, $C = 0$, so the solution is

$$v = \frac{Ft}{M_0 - rt} - \frac{g}{M_0 - rt}\left(M_0 t - \frac{rt^2}{2}\right).$$

(b) At burnout, $M_0 - rt = M_1$, so

$$v = \frac{F(M_0 - M_1)}{rM_1} - \frac{g}{M_1}\left[M_0\left(\frac{M_0 - M_1}{r}\right) - \frac{(M_0 - M_1)^2}{2r}\right]$$

$$= \frac{F(M_0 - M_1)}{rM_1} - \frac{g(M_0^2 - M_1^2)}{2rM_1} = \frac{M_0 - M_1}{rM_1}\left[F - g\left(\frac{M_0 + M_1}{2}\right)\right]$$

is the velocity at burnout. ▲

Exercises for Section 8.6

In Exercises 1–4, solve the given differential equation by the method of this section.

1. $\dfrac{dy}{dx} = \dfrac{-y}{1-x} + \dfrac{2}{1-x} + 3.$
2. $\dfrac{dy}{dx} = y\sin x - 2\sin x.$
3. $\dfrac{dy}{dx} = x^3 y - x^3.$
4. $x\left(\dfrac{dy}{dx}\right) = y + x\ln x.$

In Exercises 5–8, solve the given equation with the stated conditions.

5. $y' = y\cos x + 2\cos x, y(0) = 0.$
6. $y' = \dfrac{y}{x} + x, y(1) = 1.$
7. $xy' = e^x - y, y(1) = 0.$
8. $y' = y + \cos 5x, y(0) = 0.$
9. Rework Example 3 assuming that the voltage is $E = E_0 \sin\omega t + E_1$, i.e., a sinusoidal plus a constant voltage.
10. In Example 3, use the method on p. 373 of Section 8.1 to determine the amplitude and phase of the oscillatory part:

$$\frac{E_0}{L}\cdot\frac{1}{(R/L)^2 + \omega^2}\left\{\frac{R}{L}\sin\omega t - \omega\cos\omega t\right\}.$$

11. The equation $R(dI/dt) + (I/C) = E$ describes the current I in a circuit containing a resistor with resistance R (a constant) and a capacitor with capacitance C (a constant) as shown in Fig. 8.6.4. If $E = E_0$, a constant, and $I = 0$ at $t = 0$, find I as a function of time. Discuss your solution.

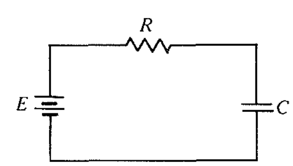


Figure 8.6.4. A resistor and a capacitor in an electric circuit.

12. Repeat Exercise 11 for the case $I = I_0$ at $t = 0$.
13. In Example 4, show that the lake will reach 90% of its limiting pollution value within 3.33 months.
14. *Mixing problem.* A lake contains 5×10^8 liters of water, into which is dissolved 10^4 kilograms of salt at $t = 0$. Water flows into the lake at a rate of 100 liters per second and contains 1% salt; water flows out of the lake at the same rate. Find the amount of salt in the lake as a function of time. When is 90% of the limiting amount of salt reached?

15. *Two-stage mixing*. Suppose the pollutants from the lake in Example 4 empty into a second smaller lake rather than the stream. At $t = 0$, the smaller lake contains 10^7 liters of pure water. The second lake has an exit stream carrying the same volume of fluid as entered. Find the amount of pollution in the second lake as a function of time.
16. Repeat Exercise 15 with the size and flow rates of the lake in Example 4 replaced by those in Exercise 14.
17. The terminal speed of a person in free fall in air is about 64 meters per second (see Example 5). How long does it take to reach 90% of terminal speed? How far has the person fallen in this time? ($g = 9.8$ meters per second^2).
18. The terminal speed of a person falling with a parachute is about 6.3 meters per second (see Example 5). How long does it take to reach 90% of terminal speed? How far has the person fallen in this time? ($g = 9.8$ meters per second^2).
19. *Falling object with drag resistance*. Redo Example 5 assuming that the resistance is proportional to the square of the velocity.
20. An object of mass 10 kilograms is dropped from a balloon. The force of air resistance is $0.07v$, where v is the velocity. What is the object's velocity as a function of time? How far has the object traveled before it is within 10% of its terminal velocity?
21. What is the acceleration of the rocket in Example 6 just before burnout?
22. How high is the rocket in Example 6 at burnout?
23. Sketch the direction field for the equation $dy/dx = y + 2x$ and solve the equation.
24. Sketch the direction field for the equation $dy/dx = -3y + x$ and solve the equation.

★25. Assuming that $P(x)$ and $Q(x)$ are continuous functions of x, prove that the problem $y' = P(x)y + Q(x)$, $y(0) = y_0$ has *exactly* one solution.

26. Express the solution of the equation $y' = xy + 1$, $y(0) = 1$ in terms of an integral.

★27. *Bernoulli's equation*. This equation has the form $dy/dx = P(x)y + Q(x)y^n$, $n = 2, 3, 4, \ldots$.
   (a) Show that the equation satisfied by $w = y^{1-n}$ is linear in w.
   (b) Use (a) to solve the equation $x(dy/dx) = x^4y^3 - y$.

★28. *Riccati equation*. This equation is $dy/dx = P(x) + Q(x)y + R(x)y^2$.
   (a) Let $y_1(x)$ be a known solution (found by inspection). Show that the general solution is $y(x) = y_1(x) + w(x)$, where w satisfies the Bernoulli equation (see Exercise 27) $dw/dx = [Q(x) + 2R(x)y_1(x)]w + R(x)w^2$.
   (b) Use (a) to solve the Riccati equation $y' = y/x + x^3y^2 - x^5$, taking $y_1(x) = x$.

★29. Redo the rocket propulsion Example 6 adding air resistance proportional to velocity.

★30. Solve the equation $dy/dt = -\lambda y + r$, where λ and r are constants. Write a two page report on how this equation was used to study the Van Meegeren art forgeries which were done during World War II. (See M. Braun, *Differential Equations and their Applications*, Third Edition, Springer-Verlag, New York (1983), Section 1.3).

Review Exercises for Chapter 8

Solve the differential equations with the given conditions in Exercises 1–22.

1. $\dfrac{dy}{dt} = 3y$, $y(0) = 1$.
2. $\dfrac{dy}{dt} = y$, $y(0) = 1$.
3. $\dfrac{d^2y}{dt^2} + 3y = 0$, $y(0) = 0$, $y'(0) = 1$.
4. $\dfrac{d^2y}{dt^2} + 9y = 0$, $y(0) = 2$; $y'(0) = 0$.
5. $\dfrac{dy}{dt} = 3y + 1$, $y(0) = 1$.
6. $\dfrac{dy}{dt} = (\cos t)y + \cos t$, $y(0) = 0$.
7. $\dfrac{dy}{dt} = t^3y^2$, $y(0) = 1$.
8. $\dfrac{dy}{dt} + 10y = 0$, $y(0) = 1$.
9. $f' = 4f$; $f(0) = 1$.
10. $f' = -4f$, $f(0) = 1$.
11. $f'' = 4f$; $f(0) = 1$, $f'(0) = 1$.
12. $f'' = -4f$; $f(0) = 1$, $f'(0) = 1$.
13. $\dfrac{d^2x}{dt^2} + x = 0$; $x = 1$ when $t = 0$, $x = 0$ when $t = \pi/4$.
14. $\dfrac{d^2x}{dt^2} + 6x = 0$; $x = 1$ when $t = 0$, $x = 6$ when $t = 1$.
15. $\dfrac{d^2x}{dt^2} - 9x = 0$, $x(0) = 0$, $x'(0) = 1$.
16. $\dfrac{d^2x}{dt^2} + 16x = 0$, $x(0) = 1$, $x'(0) = -1$.
17. $\dfrac{dy}{dx} = e^{x+y}$, $y(0) = 1$.
18. $\dfrac{dy}{dt} = 8e^{3y}$, $y(0) = 1$.
19. $\dfrac{dx}{dt} = -4x$; $x = 1$ when $t = 0$.

20. $\dfrac{dy}{dx} = y/\ln y,\ y(1) = e.$
21. $\dfrac{dy}{dt} = \dfrac{y}{t-1} + \dfrac{1}{t-1},\ y(0) = 0.$
22. $\dfrac{dx}{dt} = \dfrac{x}{8-t} + \dfrac{3}{8-t},\ x(0) = 1.$
23. Solve for $g(t)$: $3(d^2/dt^2)g(t) = -7g(t)$, $g(0) = 1$, $g'(0) = -2$. Find the amplitude and phase of $g(t)$. Sketch.
24. Solve for $z = f(t)$: $d^2z/dt^2 + 5z = 0$, $f(0) = -3$, $f'(0) = 4$. Find the amplitude and phase of $f(t)$. Sketch.
25. Sketch the graph of y as a function of x if $-d^2y/dx^2 = 2y$; $y = \frac{1}{2}$, and $dy/dx = \frac{1}{2}$ when $x = 0$.
26. Sketch the solution of $d^2x/dt^2 + 9x = 0$, where $x(0) = 1$, $x'(0) = 0$.
27. Sketch the graph of the solution of $y' = -8y$, $y = 1$ when $x = 0$.
28. Sketch the graph of $y = f(x)$ if $f' = 2f + 3$, and $f(0) = 0$.
29. Sketch the graph of the solution to $dx/dt = -x + 3$, $x(0) = 0$ and compute $\lim_{t\to\infty} x(t)$.
30. Sketch the graph of the solution to $dx/dt = -2x + 2$, $x(0) = 0$ and compute $\lim_{t\to\infty} x(t)$.
31. Solve $d^2x/dt^2 = dx/dt$, $x(0) = 1$, $x'(0) = 1$ by letting $y = dx/dt$.
32. Solve $d^2x/dt^2 = [1/(1+t)]dx/dt$, $x(0) = 1$, $x'(0) = 1$ by letting $y = dx/dt$.
33. Solve $d^2y/dx^2 + dy/dx = x$, $y(0) = 0$, $y'(0) = 1$. [*Hint*: Let $w = dy/dx$].
34. Solve $y'' + 3yy' = 0$ for $y(x)$ if $y(0) = 1$, $y'(0) = 2$. [*Hint*: $yy' = (y^2/2)'$.]
35. Solve $d^2y/dx^2 = 25y$, $y(0) = 0$, $y(1) = 1$.
36. Solve $d^2y/dx^2 = 36y$, $y(0) = 1$, $y(1) = 0$.

Differentiate the functions in Exercises 37–44.

37. $\sinh(3x^2)$
38. $\tanh(x^3 + x)$
39. $\cosh^{-1}(x^2 + 1)$
40. $\tanh^{-1}(x^4 - 1)$
41. $(\sinh^{-1}x)(\cosh 3x)$
42. $(\cosh^{-1}3x)(\tanh x^2)$
43. $\exp(1 - \cosh^{-1}(3x))$
44. $3(\cosh^{-1}(5x^2) + 1)$

Calculate the integrals in Exercise 45–50.

45. $\displaystyle\int \frac{\cosh x}{1 + \sinh^2 x}\,dx$
46. $\displaystyle\int \operatorname{sech}^2 x \tanh x \sqrt{2 + \tanh^2 x}\,dx$
47. $\displaystyle\int \frac{dx}{9 - x^2}$
48. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4}}$
49. $\displaystyle\int x \sinh x\,dx$
50. $\displaystyle\int x \cosh x\,dx$
51. A weight of 5 grams hangs on a spring with spring constant $k = 2.1$. Find the displacement $x(t)$ of the mass if $x(0) = 1$ and $x'(0) = 0$.
52. A weight hanging on a spring oscillates with a frequency of two cycles per second. Find the displacement $x(t)$ of the mass if $x(0) = 1$ and $x'(0) = 0$.
53. An observer sees a weight of 10 grams on a spring undergoing the motion $x(t) = 10\sin(8t)$.
(a) What is the spring constant?
(b) What force acts on the weight at $t = \pi/16$?
54. A 3-foot metal rod is suspended horizontally from a spring, as shown in Fig. 8.R.1. The rod bobs up and down around the equilibrium point, 5 feet from the ground, and an amplitude of 1 foot and a frequency of two bobs per second. What is the maximum length of its shadow? How fast is its shadow changing length when the rod passes the middle of its bob?

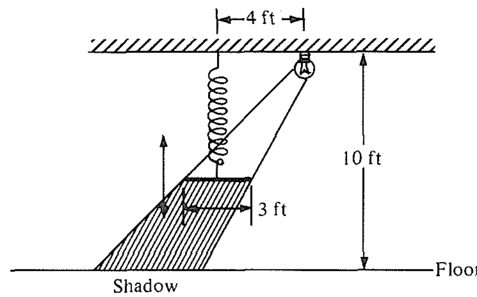


Figure 8.R.1. Study the movement of the shadow of the bobbing rod.

55. *Simple Harmonic Motion with Damping*. Consider the equation $x'' + 2\beta x' + \omega^2 x = 0$, where $0 < \beta < \omega$. (a) Show that $y = e^{\beta t}x$ satisfies a harmonic oscillator equation. (b) Show that the solution is of the form $x = e^{-\beta t}(A\cos\omega_1 t + B\sin\omega_1 t)$ where $\omega_1 = \sqrt{\omega^2 - \beta^2}$, and A and B are constants. (c) Solve $x'' + 2x' + 4x = 0$; $x(0) = 1$, $x'(0) = 0$, and sketch.
56. *Forced oscillations.*
(a) Show that a solution of the differential equation $x'' + 2\beta x' + \omega^2 x = f_0 \cos\omega_0 t$ is

$$x_1(t) = \frac{f_0}{(\omega^2 - \omega_0^2)^2 + 4\omega_0^2\beta^2}\left[2\omega_0\beta\sin\omega_0 t + (\omega^2 - \omega_0^2)\cos\omega_0 t\right].$$

(b) Show that the general solution is $x(t) = x_1(t) + x_0(t)$ where x_0 is the solution found in Exercise 55.
(c) *Resonance.* Show that the "amplitude" $f_0/[(\omega^2 - \omega_0^2)^2 + 4\omega_0^2\beta^2]$ of the solution is largest when ω_0 is near ω (the natural frequency) for β (the friction constant) small, by maximizing the amplitude for fixed f_0, ω,

β and variable ω_0. (This is the phenomenon responsible for the Tacoma bridge disaster . . . somewhat simplified of course; see Section 12.7 for further information.)

57. If a population doubles every 10 years and is now 100,000, how long will it take to reach 10 million?
58. The half-life of a certain radioactive substance is 15,500 years. What percentage will have decayed after 50,000 years?
59. A certain radioactive substance decreases at a rate of 0.00128% per year. What is its half-life?
60. (a) The population of the United States in 1900 was about 76 million; in 1910, about 92 million. Assume that the population growth is uniform, so $f(t) = e^{\gamma t}f(0)$, t in years after 1900. (i) Show that $\gamma \approx 0.0191$. (ii) What would have been a reasonable prediction for the population in 1960? In 1970? (iii) At this rate, how long does it take for the population to double?
(b) The actual U.S. population in 1960 was about 179 million and in 1970 about 203 million. (i) By what fraction did the "growth rate factor" ("γ") change between 1900–1910 and 1960–1970? (ii) Compare the percentage increase in population from 1900 to 1910 with the percentage increase from 1960 to 1970.
61. If an object cools from 100°C to 80°C in an environment of 18°C in 8 minutes, how long will it take to cool from 100° to 50°C?
62. "Suppose the pharaohs had built nuclear energy plants. They might have elected to store the resulting radioactive wastes inside the pyramids they built. Not a bad solution, considering how well the pyramids have lasted. But plutonium-239 stored in the oldest of them—some 4600 years ago—would today still exhibit 88 percent of its initial radioactivity." (see G. Hardin, "The Fallibility Factor," *Skeptic* **14** (1976): 12.)
(a) What is the half-life of plutonium?
(b) How long will it take for plutonium stored today to have only 1% of its present radioactivity? How long for $\frac{1}{1000}$?
63. (a) The oil consumption rate satisfies the equation $C(t) = C_0e^{rt}$, where C_0 is the consumption rate at $t = 0$ (number of barrels per year) and r is a constant. If the consumption rate is $C_0 = 2.5 \times 10^{10}$ barrels per year in 1976 and $r = 0.06$, how long will it take before 2×10^{12} barrels (the total world's supply) are used up?
(b) As the fuel is almost used up, the prices will probably skyrocket and other sources of energy will be turned to. Let $S(t)$ be the supply left at time t. Assume that $dS/dt = -\alpha S$, where α is a constant (the panic factor). Find $S(t)$.
64. (a) A bank advertises "5% interest on savings—but you earn more because it is compounded continuously." The formula for computing the amount $M(t)$ of money in an account t days after $M(0)$ dollars is deposited (and left untouched) is $M(t) = M(0)e^{0.05t/365}$. What is the percentage increase on an amount $M(0)$ left untouched for 1 year?
(b) A bank wants to compute its interest by the method in part (a), but it wants to give only \$5 interest on each \$100 that is left untouched for 1 year. How must it change the formula for that to occur?
65. A certain electric circuit is governed by the equation $L dI/dt + RI = E$, where E, R, and L are constants. Graph the solution if $I_0 < E/R$.
66. If a savings account containing P dollars grows at a rate $dP/dt = rP + W$ (interest with continuous deposits), find P in terms of its value P_0 at $t = 0$.
67. (a) Sketch the direction field for the equation $y' = -9x/y$. Solve the equation exactly.
(b) Find the orthogonal trajectories for the solutions in (a).
68. Consider the predator–prey model in Example 5, Section 8.5. Solve this explicitly if $r = 0$ (i.e., ignore deaths of the prey).
69. Suppose your car radiator holds 4 gallons of fluid two thirds of which is water and one third is old antifreeze. The mixture begins flowing out at a rate of $\frac{1}{2}$ gallon per minute while fresh water is added at the same rate. How long does it take for the mixture to be 95% fresh water? Is it faster to wait until the radiator has drained before adding fresh water?
70. An object falling freely with air resistance has a terminal speed of 20 meters per sec. Find a formula for its velocity as a function of time.
71. The current I in a certain electric circuit is governed by $dI/dt = -3I + 2\sin(\pi t)$, and $I = 1$ at $t = 0$. Find the solution.
72. In Example 6, Section 8.6, suppose that the burnout mass is 10% of the initial mass M_0, the burnout time is 3 minutes, and the rocket thrust is $3M_0g$. Calculate the acceleration of the rocket just before burnout in terms of M_0 and g.
73. Sketch the direction field for the equation $y' = 3y + 4$ and solve it.
74. Sketch the direction field for the equation $y' = -4y + 1$ and solve it.
75. Let $x(t)$ be the solution of $dx/dt = x^2 - 5x + 4$, $x(0) = 3$. Find $\lim_{t\to\infty} x(t)$.
76. Let x satisfy $dx/dt = x^3 - 4x^2 + 3x$, $x(0) = 2$. Find $\lim_{t\to\infty} x(t)$.
77. Test the accuracy of the Euler method by using a ten-step Euler method on the problem of finding $y(1)$ if $y' = y$ and $y(0) = 1$. Compare your answer with the exact solution and with a twenty-step Euler method.

78. Solve $y' = \csc y$ approximately for $0 \leqslant x \leqslant 1$ with $y(0) = 1$ using a ten-step Euler method.

79. Numerically solve for $y(2)$ if $y' = y^2$ and $y(0) = 1$, using a twenty-step Euler method. Do you detect some numerical trouble? What do you think is going wrong?

80. Solve for $y(1)$ if $y' = \cos(x + y)$ and $y(0) = 0$, using a ten-step Euler method.

81. Solve $y' = ay + b$, given constants a and b, by
(a) introducing $w = y + b/a$ and a differential equation for w;
(b) treating it as a separable equation; and
(c) treating it as a linear equation.
Are your answers the same?

★82. Let $x(t)$ be the solution of $dx/dt = -x + 3\sin \pi t$, $x(0) = 1$. Find α, ω and θ such that $\lim_{t\to\infty}[x(t) - \alpha\cos(\omega t + \theta)] = 0$.

★83. *Simple pendulum*. The equation of motion for a simple pendulum (see Fig. 8.R.2) is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta.$$

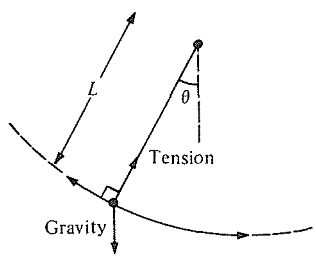


Figure 8.R.2. A simple pendulum.

(a) Let $w(t) = d\theta/dt$. Show that

$$\frac{d}{dt}\left(\frac{w^2}{2}\right) = \frac{g}{L}\frac{d}{dt}\cos\theta,$$

and so

$$\frac{w^2}{2} = \frac{g}{L}(\cos\theta - \cos\theta_0),$$

where $w = 0$ when $\theta = \theta_0$ (the maximum value of θ).

(b) Conclude that θ is implicitly determined by

$$\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \sqrt{\frac{2g}{L}}\,t.$$

(c) Show that the period of oscillation is

$$T = 2\sqrt{\frac{2L}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

where $k = \sin(\theta_0/2)$. [*Hint*: Write $\cos\theta = 1 - 2\sin^2(\theta/2)$.] The last integral is called an *elliptic integral of the first kind*, and cannot be evaluated explicitly.

(d) How does the answer in (c) compare with the prediction from linearized oscillations for θ_0 small?

★84. A photographer dips a thermometer into a developing solution to determine its temperature in degrees Centigrade. The temperature $\theta(t)$ registered by the thermometer satisfies a differential equation $d\theta/dt = -k(\theta - \bar{\theta})$, where $\bar{\theta}$ is the true temperature of the solution and k is a constant. How can the photographer determine when θ is correct to within 0.1°C?

★85. (a) Find $y = f(x)$ so that

$$\int_a^b f(x)\,dx = \int_a^b\sqrt{1 + [f'(x)]^2}\,dx$$

for all a and b.

(b) What geometric problem leads to the problem (a).